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Time Ordering and the Thermodynamics of Strange Sets: Theory and Experimental Tests

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From the spectrum of dimensions of a fractal invariant measure of a dynamical system one can extract information about the dynamical process that gave rise to the measure. This is equivalent to finding the class of Hamiltonians of an Ising model with a given thermodynamics.

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Fractal measures appear in a number of nonlinear physical phenomena like turbulence,[1-3] chaotic dynamical systems,[4-7] and fractal growth processes.[8] Such measures cannot be fully characterized by the fractal dimension of their support; rather, an infinity of generalized dimensions is called for their description.[6-8] Recently this spectrum of dimensions was linked (via Legendre transforms) to the spectrum of scaling indices of the fractal measures.[7,9]

The fractal measures that arise in dynamical systems have the particular character that they result from a time-ordered process, be it an iteration scheme or a continuous flow.[10] The resulting measures are, however, invariant to the dynamics and hence become "static" objects. Describing these invariant measures by their generalized dimensions (or spectra of scaling indices) appears, therefore, to lead to a complete loss of the dynamic information. In a sense that will be made sharper below, this description is "thermodynamic." The aim of this Letter is to demonstrate that this conclusion is in fact incorrect. With some of the provisos that are explained in the sequel, the process can be inverted, and the dynamical process that is responsible for the construction of the measure can be read from the thermodynamics. In this sense we claim that the information stored in the generalized dimensions (or spectra of scaling indices) is larger than what could be naively anticipated.

The key idea that allows such an inversion rests on the thermodynamic formalism[11-14] of dynamical systems that maps the process of refinement of the fractal measure (in time!) to a transfer matrix theory[13,14] of an appropriate Ising model. To see this analogy we begin with a partition of the set into N distinct pieces of diameters $\{l_i\}_{i=1}^{N}$. Denoting the measure of each piece by p_i, we consider the partition function

$$\Gamma(q,\tau)=\sum_{i=1}^{N}\frac{p_i^q}{l_i^\tau}. \tag{1}$$

It was argued in Ref. 7 that upon refinement of the partition, i.e., when $\max l_i \to 0$, $\Gamma(q,\tau)$ tends to infinity for $\tau>\tau(q)$ and to zero for $\tau<\tau(q)$. The quantity $\tau(q)=(q-1)D_q$, where D_q are the generalized dimensions, defined first by Renyi[15] and considered first in the context of strange attractors in Ref. 6. A convenient way of calculating $\tau(q)$ is therefore to fix $\Gamma(q,\tau)$ to a number as the partition is refined. For example, we can take

$$\Gamma(q,\tau(q))=1. \tag{2}$$

For the purpose of the present argument we consider special partitions such that $p_i=\text{const}$. If the number of boxes of the partition is denoted N_n in the nth generation of refinement, then $p_i=N_n^{-1}$. Inserting this in Eq. (2) we get

$$N_n^{q(\tau)}=\textstyle\sum_i |l_i^{(n)}|^{-\tau}, \tag{3}$$

where now the function $q(\tau)$ rather than $\tau(q)$ is employed. Typically, the number N_n grows exponentially with n. Writing

$$N_n = a^n, \qquad (4)$$

we perform now one step of the refinement of the set and consider

$$\frac{N_{n+1}^{q(\tau)}}{N_n^{q(\tau)}} = a^{q(\tau)} = \frac{\sum_i |l_i^{(n+1)}|^{-\tau}}{\sum_j |l_j^{(n)}|^{-\tau}}. \qquad (5)$$

The result of this argument will be that

$$a^{q(\tau)} = \lambda(\tau) \qquad (6)$$

is an eigenvalue of a transfer matrix.[13,14] To show this we point out that whenever $a \leq 2$ we can write the index i of $l_i^{(n)}$ as $\epsilon_n, \ldots, \epsilon_0$, where ϵ_l takes on binary values 0 or 1. For $2 \leq a \leq 3$ we need ϵ_l that takes on ternary values 0, 1, 2, etc. We thus rewrite Eq. (5) as

$$\sum_{\epsilon_{n+1},\ldots,\epsilon_0} |l(\epsilon_{n+1},\ldots,\epsilon_0)|^{-\tau} = \lambda(\tau) \sum_{\epsilon_n,\ldots,\epsilon_0} |l(\epsilon_n,\ldots,\epsilon_0)|^{-\tau}. \qquad (7)$$

Next we define the daughter-to-mother ratio, also known as the scaling function,[16] by

$$\frac{l(\epsilon_{n+1},\ldots,\epsilon_0)}{l(\epsilon_n,\ldots,\epsilon_0)} = \sigma(\epsilon_{n+1},\ldots,\epsilon_0). \qquad (8)$$

Substitution in Eq. (7) leads to

$$\sum_{\substack{\epsilon_{n+1},\ldots,\epsilon_0 \\ \epsilon_n',\ldots,\epsilon_1'}} \delta_{\epsilon_n,\epsilon_n'} \cdots \delta_{\epsilon_1,\epsilon_1'} \sigma^{-\tau}(\epsilon_{n+1},\ldots,\epsilon_0) |l(\epsilon_n',\ldots,\epsilon_1',\epsilon_0)|^{-\tau} = \lambda(\tau) \sum_{\epsilon_n,\ldots,\epsilon_0} |l(\epsilon_n,\ldots,\epsilon_0)|^{-\tau}, \qquad (9)$$

where upon substitution we also added summations on $\epsilon_1', \ldots, \epsilon_n'$ which were immediately compensated by the Kronecker δ's. We therefore conclude that by defining a transfer matrix T,

$$T_{(\epsilon_{n+1},\ldots,\epsilon_1)(\epsilon_n',\ldots,\epsilon_1',\epsilon_0)} = \sigma^{-\tau}(\epsilon_{n+1},\ldots,\epsilon_0)\delta_{\epsilon_n,\epsilon_n'} \cdots \delta_{\epsilon_1,\epsilon_1'}, \qquad (10)$$

we obtain the result that $\lambda(\tau)$ is an eigenvalue of T.

It should be clear by now how the problem maps onto an Ising model. The number of spin states depends on ϵ being binary, ternary, quaternary, etc. The range of interaction depends on how far back the memory goes in determining daughter-to-mother ratios. If we can truncate $\epsilon_n, \ldots, \epsilon_0$ after, say, ϵ_{n-r}, then we have r nearest-neighbor interactions. In that case $\lambda(\tau)$ becomes the largest eigenvalue of T, because (9) can be iterated.

In general, sets arising in dynamical systems might or might not have long-range interactions. We know however that in sets that belong to the borderline of chaos the memory usually falls off exponentially. We therefore limit ourselves to such sets, and ask if we can find the dynamics from the information about generalized dimensions D_q, spectra of scaling indices $f(\alpha)$, etc.[7,9] In other words, given $q(\tau)$, can we find the winding number, the type of dynamical system (quadratic maximum, cubic inflection point, etc.), and the elements of the scaling function? We shall see that the first two questions are answered positively in full and the last one in part.

To focus the ideas let us examine two cases of sets that have binary ϵ. The first is the 2^∞ cycle at the accumulation point of period doubling, and the second is the critical orbit with irrational winding number $w_G = 1.618\ldots$ (the golden mean), at the point of breakdown of a two-torus. For the 2^∞ cycle, $a = 2$ and the binary tree is complete. Therefore all the elements of the transfer matrix are nonzero. For the golden-mean cycle, $a = w_G$ (asymptotically) and the tree cannot be complete. Therefore there must be zero elements in T. To see which are the zero elements we use the fact that for these sets at the onset of chaos the orbits are time ordered exactly such as to match the ϵ ordering of the partitions. If we expand the time in Fibonacci numbers F_n, $F_{n+1} = F_n + F_{n-1}$, $F_0 = F_1 = 1$, i.e.,

$$t = \sum \epsilon_n F_n, \qquad (11)$$

we discover immediately that adjacent 1's are not allowed since they cause overflow of the time axis. Thus $\sigma(1,1)$, for example, must be zero. Accordingly, the lowest-order nontrivial approximation of the transfer matrix reads

$$T_{w_G}^{(1)} = \begin{pmatrix} \sigma_{00}^{-\tau} & \sigma_{01}^{-\tau} \\ \sigma_{10}^{-\tau} & 0 \end{pmatrix}, \qquad (12)$$

whereas the period-doubling case has a nonzero σ_{11}. In a similar way, if we consider period tripling versus, say, a critical orbit with silver-mean winding number, $w_S = 1 + \sqrt{2}$, we deal with cases that have ternary ϵ's. For the period-tripling case, $a = 3$ and the ternary tree is complete. For silver-mean trajectories neither consecutive 2's nor the combinations $\ldots$, ϵ_l, 1, 2,

ϵ_j, ... are allowed, and so on.

A strategy for the extraction of dynamic information therefore suggests itself. Given $q(\tau)$ we begin by attempting to fit an equation $a^{q(\tau)} = \lambda(\tau)$, where $\lambda(\tau)$ is calculated from a 2×2 matrix. Writing the general characteristic polynomial

$$\lambda^2(\tau) - \lambda(\tau)(\sigma_{00}^{-\tau} + \sigma_{11}^{-\tau}) + (\sigma_{00}\sigma_{11})^{-\tau} - (\sigma_{01}\sigma_{10})^{-\tau} = 0, \qquad (13)$$

we see that σ_{01} and σ_{10} appear only as a product,[13] and thus $\lambda(\tau)$ depends on three scales, $\sigma_{00} = s_1$, $\sigma_{11} = s_2$, $\sigma_{10}\sigma_{01} = s_3$. We can further use our knowledge of $q(\tau)$, and in particular the knowledge of D_0 (which is the fractal dimension) in Eq. (6), which for $q = 0$ reads

$$1 = \lambda(-D_0). \qquad (14)$$

This can be used in Eq. (13) to eliminate one of the three scales, leaving (6) as an equation with three unknowns, i.e., two scales and the number a. We solved such equations by a multidimensional Newton-Raphson technique, that proved to be rapidly convergent.

To demonstrate how the procedure works we chose not to employ theoretical data that are perfectly accurate, but rather use experimental data that are subject to some uncertainty. As case models we chose the data pertaining to a golden-mean orbit and to the period-doubling scenario in a forced Rayleigh-Bénard experiment using mercury as the fluid.[17] The experiment and its results were reported elsewhere,[9,18] and here we summarize in Table I the $q(\tau)$ values that were obtained from the experimental orbits [we have mainly used positive values of q which typically lead to smaller uncertainties in $\tau(q)$].

Picking any three values of τ from the data we can solve Eqs. (6), (13) (with $D_0 = 1$), and (14) numerically. Table II shows typical results. For the golden-mean data the use of any three values of τ that are reasonably accurate leads to a very rapid convergence to $\sigma_{11} = 0$ and the number a being very close to the golden mean. Taking this as a strong indication that

TABLE I. The experimentally obtained numbers $\tau(q)$ from a forced Rayleigh-Bénard system (Refs. 9 and 18).

Golden mean		Period doubling	
q	$\tau(q)$	q	$\tau(q)$
0.3	−0.686	1.3	0.1557
0.6	−0.384	1.6	0.3072
0.9	−0.093	1.9	0.4563
1.2	0.182	2.2	0.6036
1.5	0.445	2.5	0.7515
1.8	0.696	2.8	0.8964
2.1	0.935	3.1	1.0416
2.4	1.162	3.4	1.1856
2.7	1.377	3.7	1.3257
−0.3	−1.339	4.0	1.464
−0.6	−1.696	4.3	1.5972
−1.8	−3.276	4.6	1.728
		4.9	1.8486
		5.2	1.9614

we have indeed a golden-mean orbit, we can now substantiate this by going to the next order where the transfer matrix reads

$$T_{w_G}^{(2)} = \begin{pmatrix} \sigma_{000}^{-\tau} & \sigma_{001}^{-\tau} & 0 & 0 \\ 0 & 0 & \sigma_{010}^{-\tau} & 0 \\ \sigma_{100}^{-\tau} & \sigma_{101}^{-\tau} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \qquad (15)$$

A calculation shows that there are only two independent scales in (15). Assuming then that $a = 1.618$, we can use essentially any two entries from Table I to solve for these scales. We get a wonderful fit with $\sigma_{000} = 0.44 \pm 0.03$ and $\sigma_{000}\sigma_{010}\sigma_{101} = 0.26 \pm 0.03$. We consider this excellent support of the ideas presented here.

The period-doubling data yield similarly satisfactory results. Here we use the experimental value of $D_0 \approx 0.54$ [i.e., $1 = \lambda(-D_0 = -0.54)$], and solve Eq. (6) in the lowest-order nontrivial case of a 2×2 transfer matrix. Typical results are summarized in

TABLE II. Typical results of the inversion of the data in Table I. Other values of $\tau(q)$ give similar results.

Golden mean 2×2 matrix				Golden mean 4×4 matrix			Period doubling 2×2 matrix			
Values of q	a	σ_{00}	σ_{11}	Values of q	σ_{000}	$\sigma_{000}\sigma_{010}\sigma_{101}$	Values of q	a	σ_{00}	σ_{11}
0.3, 0.6, 0.9	1.618	0.467	-1.1×10^{-11}	0.3, 0.6	0.456	0.261	1.3, 1.6, 1.6	2.003	0.441	0.194
0.3, 0.9, 1.2	1.619	0.469	3.8×10^{-82}	0.9, 1.2	0.468	0.227	1.3, 2.8, 4.9	1.999	0.379	0.176
0.3, 0.9, 1.5	1.619	0.469	3.7×10^{-72}	1.2, 1.5	0.450	0.252	1.3, 1.9, 5.2	2.002	0.405	0.174
0.6, 0.9, 1.2	1.619	0.471	1.3×10^{-73}	1.5, 2.1	0.441	0.269	1.6, 2.2, 4.9	2.000	0.409	0.180
0.9, 2.1, 2.4	1.607	0.344	1.1×10^{-54}	−0.3, −0.6	0.425	0.255	1.9, 2.2, 4.9	1.998	0.400	0.179
0.3, −0.3, −0.6	1.652	0.457	5.8×10^{-3}	−0.6, −1.8	0.419	0.259	1.9, 2.5, 4.3	1.992	0.391	0.186

Table II. Notice that $a=2$ and no element of T is zero. The only dynamics at the borderline of chaos consistent with this is that of an infinitely doubled orbit.

Notice that we get good information about the underlying dynamical system as well; in the case of period doubling the scales obtained are a number and its square. This indicates a map with a quadratic maximum. In the golden-mean case we get from the fit to the matrix (15) values for σ_{000} and for $\sigma_{000}\sigma_{010}\sigma_{101}$. Comparing $s_1=\sigma_{000}$ to $s_2=\sigma_{010}\sigma_{101}$ we find $s_1=s_2^{3/2}$. This is consistent with $\sigma_{010}\approx\sigma_{101}\approx\alpha^{-1}$ and $\sigma_{000}\approx\alpha^{-3}$ ($\alpha=1.2558\ldots$ is a universal number[19]) which is a strong indication for a map with a cubic inflection point.

If the data were not consistent with winding number ≤ 2, a fit to a 2×2 matrix would have failed.[20] In that case one should try a fit to a ternary tree. Then the lowest-order nontrivial matrix is of size 3×3. Similar constraints on $\lambda(-D_0)$ can be used to reduce the number of free parameters. Obviously if no good fit is obtained with ternary trees, one can seek solutions with quaternary trees, etc., but the number of free scales increases and numerical convergence becomes a tedious affair.

In summary, we show that much of the dynamical information can be retrieved from data which appear "static." In some sense this is like retrieving whole potatoes from mashed potatoes.[21] We can get the winding number, and the nature of the underlying dynamical system. We cannot retrieve the full scaling functions. There are scales that always appear in products and in this sense there is degeneracy in the "thermodynamic" description.[13] We can find, however, to what *class* the scaling function belongs, and this appears sufficient to pinpoint the dynamics (at least in the class of dynamics at the borderline of chaos).

It is our feeling that the mapping onto transfer matrix language is extremely useful for the study of strange sets, and attempts to generalize it off the borderline of chaos will be reported elsewhere.

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20One should remark that Eq. (6) for $q=1$ reads $a=\lambda(0)$. For a 2×2 matrix, this equation allows only two solutions, $a=w_G$ and $a=2$. A winding number $1<w<2$ that is not the golden mean would lead to $q(\tau)$ data that would fail to fit either a 2×2 full matrix or a 2×2 matrix with $\sigma_{11}=0$, especially for q around 1. It is therefore important to use q values such that at least one q value is chosen such that $|q|\leq 1$ in the first attempt to fit a 2×2 matrix. High q values lead to spurious fits to $a=2$.

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