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Roads to turbulence in dissipative dynamical systems

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Three scenarios leading to turbulence in theory and experiment are outlined. The respective mathematical theories are explained and compared.

CONTENTS

I. INTRODUCTION

Every physicist is exposed early in his career to solvable dynamical problems, for example, the harmonic oscillator and the Kepler problem. One also learns that a damped pendulum reaches its equilibrium position, and one learns how to find the exponential functions describing the approach to this equilibrium. Quite soon, one becomes aware that not all dynamical problems are explicitly solvable, even allowing for solutions in terms of the more complicated transcendental functions. This situation may occur for systems with few degrees of freedom, (i.e., few dynamical variables), and without external noise. In addition, it is not restricted to Hamiltonian problems, but appears as well for dynamical systems with internal friction, called dissipative dynamical systems. The reason for this difficulty is the fact that *dynamical problems with regular equations may have solutions which behave irregularly in time*.

We would like to understand, in the absence of explicit solutions, more about the qualitative aspects of these irregular solutions. There is no general classification of dynamical systems which is sufficiently fine to account for all possible types of erratic behavior of their solutions, and even such simple systems as a forced pendulum with friction are exceedingly hard to analyze. One would nevertheless like to find similarities among, and predictions for, various dynamical systems.

The aim here is to present an approach to the understanding of irregular (or nearly irregular) phenomena, which has been relatively successful recently.[1] To avoid any misunderstanding, I must insist that this approach does not reach any conclusions about such matters as the beautiful turbulences on Jupiter or the dynamics of the Niagara falls.[2] Rather, by setting more modest aims, I describe here examples of relatively simple, but nevertheless aperiodic behavior, and put them in perspective. In this view, systems exhibiting this behavior are still sufficiently irregular to be called turbulent, and in fact some of their aspects are found in (irregular) convection of fluids. All forms of aperiodicity (even very weak ones) are of interest, but the words *aperiodic*, *erratic*, *chaotic*, and (*weakly*) *turbulent* will be used interchangeably for any of these forms.

The approach I describe has its roots in the general study of deterministic differential equations which are supposed to model the physical (chemical, ...) system under investigation (Smale, 1967). Throughout, we shall suppose that the system depends on an external controllable parameter and that for some value of the parameter its dynamical behavior is well understood (e.g., the system could have only a stable equilibrium state, or a stationary solution). As the parameter is changed from this value, the qualitative behavior of the system may change, too. After a finite or infinite succession of such changes the system may present erratic behavior in the sense that its time evolution may be quite unpredictable on large time scales, or it may show broad-band spectrum or may not be periodic any more. Some systems may show features of a stochastic process,[3] although no external noise source is present in the dynamical equations.

II. DISSIPATIVE SYSTEMS AND THEIR ATTRACTORS

In order to describe our main topic, we need an adequate language for describing deterministic evolution equations. Typical behavior will be described in terms of the attractors of a system. The evolution equations, for fixed value of the parameter, will be assumed

[1] In a way, this approach can be viewed as a concretization of some aspects of Thom's (1972) catastrophe theory.

[2] For a discussion of "fully developed turbulence," see, for example, Monin and Yaglom (1975).

[3] Good expository references about these aspects are Bowen (1975) and Lanford (1978).

[thro]ughout to be of one of two types, namely,

$$\frac{d}{dt}\mathbf{x}(t)=\mathbf{F}(\mathbf{x}(t)), \tag{1}$$

or

$$\mathbf{x}_{n+1}=\mathbf{F}(\mathbf{x}_n). \tag{2}$$

Here $\mathbf{x}$ is a vector in R^m, $m \geq 1$ and each of its components describes a "mode" or a coordinate. When $\mathbf{F}$ will depend on a parameter, we shall denote it by μ and write $\mathbf{F}_\mu$. Typical examples of dynamical systems of the form of Eq. (1) are listed in Table I.

We shall describe later how Eq. (2) appears naturally in applications; in any case, the simple dynamical system (discrete iteration) which is defined by

$$x_{n+1}=f(x_n),$$

where $x_n \in R$, $n=0,1,2,\ldots$ and $f: R \to R$ is continuous, often serves as a guiding tool (Collet and Eckmann, 1980). Here, one should think of n as the (discrete) time.

It is well known that in Hamiltonian dynamics Liouville's theorem asserts that the flow $t\to\mathbf{x}(t)$ preserves volumes in phase space. If we denote by $\mathbf{x}(\mathbf{y},t)$ the solution of Eq. (1) with initial condition $\mathbf{x}(\mathbf{y},t=0)=\mathbf{y}$, and if

$$\sum_{i=1}^{m}\frac{\partial F_i}{\partial x_i}(\mathbf{x})=0,$$

then the flow preserves volumes locally. On the other hand, for systems with internal friction, called *dissipative systems*, such as the last three examples in Table I, the flow contracts volumes, i.e.,

$$\sum_{i=1}^{m}\frac{\partial F_i}{\partial x_i}(\mathbf{x})<0,$$

or (equivalently)

$$\sum_{i=1}^{m}\frac{\partial \dot{x}_i(\mathbf{y},t)}{\partial y_i}<0,$$

where $\dot{\mathbf{x}}=d\mathbf{x}/dt$.

We shall deal exclusively with dissipative systems, and we start now with the description of their attractors. Assume there is a finite volume V in state space (R^m) such that if $\mathbf{y}\in V$ then $T^t\mathbf{y}=\mathbf{x}(\mathbf{y},t)$ is in V for all $t>0$. Since the flow T^t decreases volumes, the sets T^tV decrease as $t\to\infty$ to a set

$$W=\bigcap_{t>0}T^tV$$

(of zero volume). Thus every solution curve starting at some $\mathbf{y}\in V$ approaches W as $t\to\infty$. We can alternately say that if $\mathbf{y}\in V\setminus W$ then $\mathbf{y}$ is *transient* and the curve $T^t\mathbf{y}$ will for some sufficiently large t definitively depart from $\mathbf{y}$ and converge to W. This is in sharp contrast with the situation encountered in nondissipative closed systems, where almost all curves $T^t\mathbf{y}$ return infinitely often arbitrarily close to their initial state $\mathbf{y}$. We shall not discuss the question of transience, although this is an interesting subject. Therefore we consider only systems which have attained some sort of "internal equilibrium." In other words, we analyze the motion on W or on parts of W, assuming the orbits which tend to W but are not in it behave similarly to those in W, at least after a sufficient lapse of time. These parts of W will be called *attractors*, and studying attractors only amounts to neglecting transient behavior. Before reading the definition of attractors, it should be kept in mind that there is no universal agreement about what the best definition should be [see, for example, Newhouse (1980b), Shub (1980), Lanford (1981)].

Definition. An *attractor* for the flow T^t is a compact set X satisfying

(1) X is invariant under T^t: $T^tX=X$.

(2) X has a shrinking neighborhood, i.e., there is an open neighborhood U of X, $U\supset X$ such that $T^tU\subset U$ for $t>0$ and $X=\bigcap_{t>0}T^tU$.

This definition excludes *repellors*—for example, an isolated fixed point $\mathbf{x}$, $T^t\mathbf{x}=\mathbf{x}$, in whose neighborhood there is for every $\varepsilon>0$ a $\mathbf{y}$ with $|\mathbf{y}-\mathbf{x}|<\varepsilon$, which escapes away from $\mathbf{x}$, i.e., $|T^t\mathbf{y}-\mathbf{x}|$ grows (relatively) large. A repellor $\mathbf{x}$ would be in W, but not in X. We are not interested in repellors, since from an experimental point of view only attractors can play a role. Many points behave like the points on attractors, but only few behave like a repellor; a repellor is a generalization of an unstable equilibrium point or of a saddle point.

A good definition of an attractor needs another ingredient which generalizes the description of k separate stable equilibria to k separate attractors. This is achieved by the following requirement.

(3) The flow T^t on X is *recurrent* and *indecomposable*. Recurrent means T^t is nowhere transient on X: If U is an open set in V and if $U\cap V\neq\varnothing$, then there are arbitrarily large values for t such that $T^t\mathbf{x}\in X\cap U$ when $\mathbf{x}\in X\cap U$. Indecomposable means that X cannot be split into two nontrivial closed invariant pieces.

In the simplest dynamical systems the situation might be as shown in Fig. 1. There are two attractors, x_1 and x_2, which are stable fixed points. There basins of attraction are respectively the left and right sides of the line L. The line L is attracted by x_3, which is not an attractor, since it also has an unstable direction. It is a saddle point. With our previous definitions, $W=\{x_1,x_2,x_3\}$.

If X is an attractor, its *basin of attraction* is defined to be the set of initial points $\mathbf{x}$ such that $T^t\mathbf{x}$ approaches X as $t\to\infty$.

TABLE I. Dynamical systems and their phase-space coordinates.

System[a]	Interpretation of coordinates
Hamiltonian mechanics	Coordinates p,q in phase space
Particle accelerators	Deviations from ideal trajectory
Hydrodynamics	Fourier modes of velocity field (not position of molecules)
Chemical reactions	Concentrations
Electrical circuits	Currents, voltages

[a]Some introductory references are Siegel and Moser (1976), Hagedorn (1957), Foias and Temam (1979), Nicolis and Prigogine (1977), and Brayton and Moser (1964).

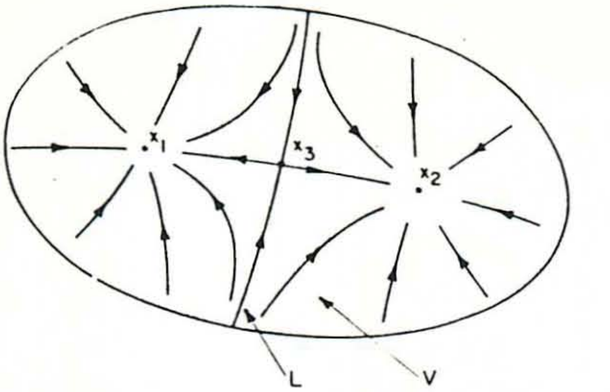

FIG. 1. Phase portrait illustrating two stable (x_1, x_2) and one unstable (x_3) fixed point.

It is now time to point out some misconceptions which could arise from the simple picture of Fig. 1.

(1) Although T^t contracts volumes, it *need not contract lengths.* If we take snapshots of T^t at $t=0,1,2,$ say, we may have the picture shown in Fig. 2(a) but could also get that of Fig. 2(b) or even that of Fig. 2(c). In particular, even if all points in V converge to a single attractor X, one still may find that points which are arbitrarily close initially may get macroscopically separated on the attractor after sufficiently large time intervals. This property is called *sensitive dependence on initial conditions.* It is *not* excluded for area-contracting flows, i.e., it can, and will, occur in dissipative dynamical systems. An attractor exhibiting this property will be called a *strange attractor.* Strange attractors are neither periodic points nor periodic orbits. Observe, however, that there exists a large variety of attractors which are neither trivial (i.e., they are neither periodic orbits nor fixed points) and which are not strange attractors. All of them seem to present more or less pronounced chaotic features. We shall call the motion on any nontrivial attractor weakly

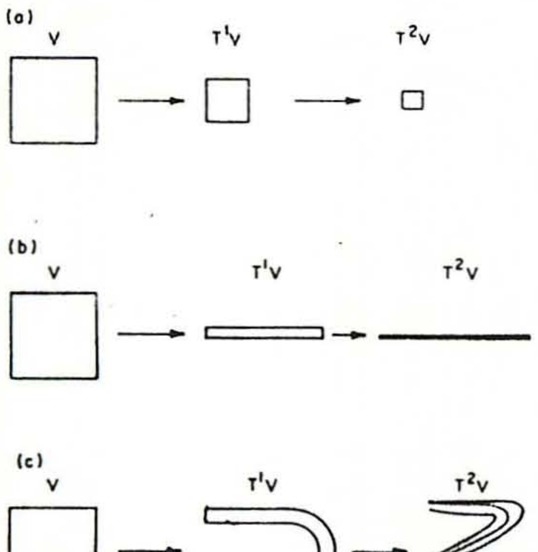

FIG. 2. (a) Contraction of volume in phase space. (b) Contraction of volume in phase space, with stretching of length. (c) Contraction of volume, stretching of length, and folding.

turbulent, erratic, etc., independently of whether or not the attractor is strange.

(2) Even simple dynamical systems may have an *infinity* of distinct attractors. As an example, it has been shown [Newhouse, 1980a; see also Levi, to appear] that the iterative scheme of Hénon

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} \to \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 1 + y_n - a x_n^2 \\ b x_n \end{pmatrix}$$

has an infinity of attractors at some values of a near 1.15357 and $b=0.3$. The attractors correspond to periodic points of higher and higher period, which may be numerically indistinguishable from a strange attractor. Incidentally, it is believed that for some values of a and b the above system does have a strange attractor, but this has not been proved so far.[4]

(3) Basins of attraction may be complicated, even if the attractors are simple. A very old example[5] is the following: Consider the map

$$z_{n+1} = z_n - \frac{z_n^3 - 1}{3 z_n^2},$$

defined on $\mathbb{C} \setminus \{0\}$. This is the Newton algorithm for finding the roots of $z^3 = 1$. It has three stable fixed points $z=1$, $\exp(i2\pi/3)$, $\exp(-i2\pi/3)$, with domains of attraction $\mathfrak{D}_1, \mathfrak{D}_2, \mathfrak{D}_3$. One can show that the boundary points of $\mathfrak{D}_1, \mathfrak{D}_2, \mathfrak{D}_3$ coincide. So these three regions must be highly interlaced.

III. THE PROBLEM OF CLASSIFYING ATTRACTORS. SCENARIOS

In the spirit of the preceding discussion, one should arrive at a description of the nontransient behavior of dynamical systems by classifying their attractors and the motion on them. This aim is clearly felt throughout the literature on dynamical systems. One is, however, far from any complete classification of attractors, or even from a canonical choice of adequate classification criteria. What I present here is a more modest approach which will lead to a *description of some nontrivial attractors, which have the additional feature that they arise as modifications of trivial attractors as an external parameter is changed.*[6] Thus, instead of considering a single problem, we deal with a one-parameter family of problems:

$$\frac{d}{dt}\mathbf{x}(t) = \mathbf{F}_\mu(\mathbf{x}(t)), \quad \mathbf{x}(0) = \mathbf{y}$$

or

$$\mathbf{x}_{n+1} = \mathbf{F}_\mu(\mathbf{x}_n), \quad \mathbf{x}_0 = \mathbf{y}.$$

The parameter μ, in the list of Table I, can be thought of as the strength of a driving force, the amount of friction, the amount of chemicals added per time unit, etc. It is assumed that μ stays fixed during the whole duration of an experiment. We are interested in the *changes of the attractors* as the parameter is varied.

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[4] A partial answer is in Misiurewicz (1980).

[5] I have heard this from F. Sergeraert.

[6] This procedure has been advocated in Ruelle and Takens (1971).

eral, the attractor changes smoothly for small ations of the parameter. For example, a fixed t may move a little bit as the parameter is varied, stable limit cycle may change its shape and/or the needed to complete a cycle (see Fig. 3).

Sometimes, however, the topological nature of the attractor may change as the parameter crosses a point μ_B. One calls this a *bifurcation* point. For example, in Fig. 4 the stable fixed point at μ_1 changes to a stable limit cycle at μ_2 (plus an unstable fixed point). Quite often a bifurcation is prompted by the crossing of eigenvalues of the linearized flow at the fixed point (or periodic orbit) through the unit circle when the parameter passes through μ_B.

A first bifurcation may be followed by further bifurcations, and we may ask what happens when a certain sequence of bifurcations has been encountered. In principle there is an infinity of further possibilities, but, in some sense to be specified, not all of them are equally probable. The more likely ones will be called *scenarios*, and below we shall examine three prominent scenarios which have had theoretical and experimental success. One should hope that further relevant scenarios will be found in the future.

We are now going to look at the nature of the prediction which can be made with the help of scenarios, since this may be a somewhat unfamiliar way of reasoning. But it appears that this kind of argument has the most promising chances of illuminating the nature of chaotic behavior. The statement of a scenario always takes the form "if... then...," i.e., if certain things happen to the attractor as the parameter is varied, then certain other things are likely to happen as the parameter is varied further. The mathematical meaning of "likely" may depend on the scenario and will be described below for each of the scenarios. But what does likely mean in a physical context? I do not intend to go to any philosophical depth but, rather, take a pragmatic stand. (1) One never knows exactly which equation (i.e., which F) is relevant for the description of a given physical system. (2) When an experiment is repeated, the equations may have slightly changed (e.g., the gravitational effects change on the earth by the motion of the moon). (3) The equation under investigation is one among several, all of which are very close to each other. (4) If among these there are many which satisfy the conclusions of the scenario, then we will say that if we perform an actual experiment, it will be probable that the conclusions of the scenario apply.

In general, a scenario deals with the description of a few attractors. On the other hand, a given dynamical system may have many attractors. Therefore, *several scenarios may evolve concurrently in different regions of phase space*. There is thus no contradiction if several scenarios occur in a given physical system, depending on how the initial state of the system is prepared. In addition, the relevant parameter ranges may overlap, and while the basins of attraction for different scenarios must be disjoint, they may be interlaced.

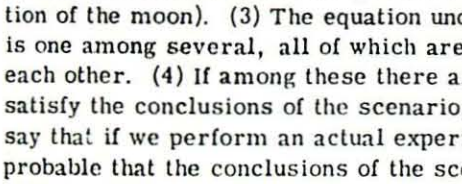


FIG. 3. Phase portraits illustrating stable limit cycles.

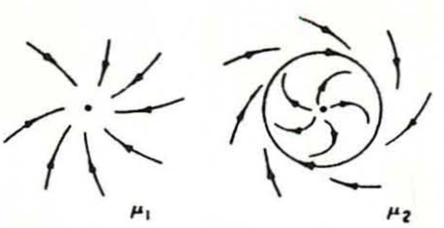


FIG. 4. Phase portraits illustrating Hopf bifurcation.

It is implicit in the preceding discussion that *a scenario does not describe its domain of applicability*. We have already stated that a scenario consists of an "if" part and a "then" part, which should be a statement that something is likely to happen. But there is no attempt being made to say how probable the "if" part is; such statements must be found by other, maybe more specific, theories. Therefore, if the hypotheses of a scenario do not apply, nothing is falsified and there is no contradiction, but no prediction is being made. Finally it should be stressed that while scenarios intend to describe roads to turbulence, no claim is made that this is the only way to find turbulence. Turbulence also occurs elsewhere, e.g., in the Niagara falls.

Let us recapitulate the main advantages and handicaps of the procedure.

(1) The turbulence described in the scenarios which have been found so far is a simple form of temporal aperiodicity, whose appearance is well under control. It has not been possible, so far, to find scenarios which lead to the rich spatiotemporal structure of fully developed turbulence, but nothing excludes in principle finding such scenarios.

(2) The theory is completely general, but it cannot describe its domain of applicability.

(3) The main field of study for scenarios is deterministic evolution equations, leading to stochastic behavior, whose occurrence *does not need any external noise source. Any external noise should be thought of as an additional complication*.[7]

The description of scenarios will be uniform, so that differences and similarities may appear more clearly. After a *mathematical description*, the scenario will be described in more simple-minded terms, followed by *interpretation, experimental evidence*, and a short description of the *influence of external noise*. Since there seems to be a general interest in such external noise, a final section will be devoted to a summary of the known results for the various scenarios. Table II at the end will summarize the results.

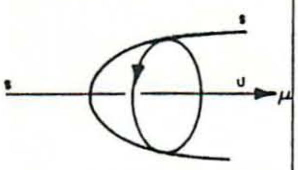
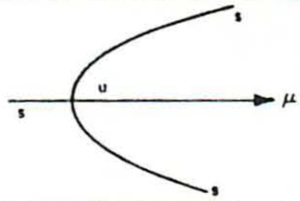
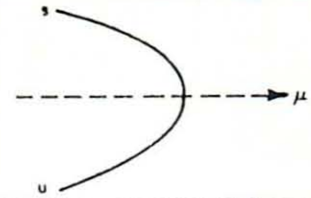
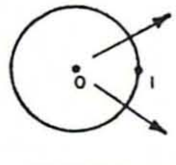
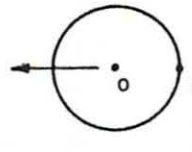
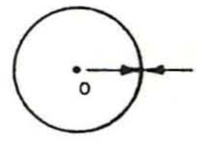
IV. THE RUELLE-TAKENS-NEWHOUSE SCENARIO

A. Description

This scenario is the oldest one, if we disregard the Landau scenario (see below for a discussion of why this

[7]For other formulations of this point of view, see Lanford (1981), Ruelle (1980), or Lorenz (1963).

TABLE II. Summary of the three scenarios discussed in this paper.

SCENARIO	Ruelle-Takens-Newhouse	Feigenbaum	Pomeau-Manneville
Typical bifurcations	Hopf	Pitchfork	(inverse) Saddle-node
Bifurcation diagram (s = stable, u = unstable).			
Eigenvalues of linearization in complex plane as μ is varied			
Main phenomenon	After 3 bifurcations strange attractor "probable"	Infinite cascade of period doublings with universal scaling of parameter values $\mu_i - \mu_\infty \sim (4.6692)^{-i}$	Intermittent transition to chaos. Laminar phase lasts $\sim(\mu-\mu_c)^{-1/2}$
Measurement	Power spectrum, correlation	Power spectrum subharmonics ~ 13.5 db below preceding level	Real-time measurements
Small noise	no influence	high periods disappear (noise level must go down by 6.62 to see one more period doubling)	time of laminarity scales as $(\mu-\mu_c)^{-1/2}\, T(\sigma/(\mu-\mu_c)^{3/4})$ for noise of standard deviation σ

is an inadequate scenario) (Ruelle and Takens, 1971).

In abstract mathematical terms, the situation is as follows.

Theorem (Newhouse, Ruelle, Takens, 1978).[8] *Let v be a constant vector field on the torus $T^n = R^n/Z^n$. If $n \geq 3$, every C^2 neighborhood of v contains a vector field v' with a strange Axiom A attractor. If $n \geq 4$, we may take C^∞ instead of C^2.*

For the definition of Axiom A vector fields, see Smale (1967).

B. Assumptions

It is now easy to describe an "if" for a scenario which implies the conditions of the theorem and hence its conclusion.

Assume a system $\dot{\mathbf{x}} = F_\mu(\mathbf{x})$ has a steady-state solution $\mathbf{x}_\mu$ for $\mu < \mu_c$. *Assume* further that this steady-state solution loses its stability through a *Hopf bifurcation* (Ruelle and Takens, 1971) (i.e., a pair of complex eigenvalues of

$$A_{ij} = \left.\frac{\partial F_\mu^{(i)}}{\partial x_j}\right|_{x = x_\mu}$$

crosses the imaginary axis, or $\exp A_{ij}$ has eigenvalues crossing the unit circle). This means that the steady state (a constant flow or an equilibrium) becomes oscillatory; we may say that some mode has been destabilized. *Assume* that this happens three times in succession, and that the three newly created modes are essentially independent [see Ruelle and Takens (1971) for details]. Thus the "if" part of the scenario is as shown in Fig. 5. Under all these assumptions, the scenario of Ruelle-Takens asserts: *A strange attractor may occur.* Its occurrence is "likely" in the following sense.

C. Interpretation

In the space of all differential equations, some equations have strange attractors; others have none. Those which do form a set which contains a subset which is *open* in the C^2 topology. The closure of this open set contains the constant vector fields on the torus T^3.

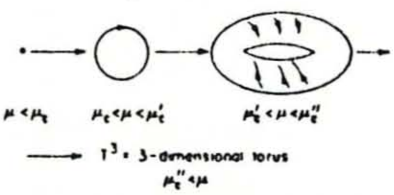
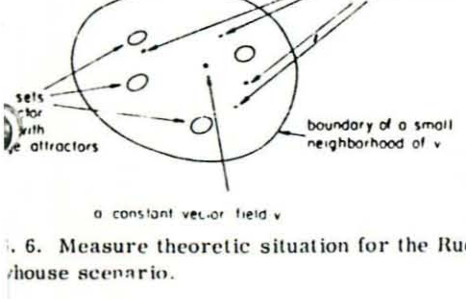


FIG. 5. Three critical values of the parameter μ_c, μ'_c, μ''_c, and the associated motion in phase space.

[8]Ruelle and Takens's original work (1971) needed four dimensions. This was reduced to three by using an idea of Plykin

a property of differential equations holds in an open /, then if we vary the coefficients of the differential ,uations sufficiently little, the property continues to . Thus the strangeness of the attractor is stable under small perturbations of the dynamical system; in other words, it is not exceptional. We can compare this with the *Landau scenario* (Landau and Lifshitz, 1959, III, Sec. 103), which assumes that the flow on the three-torus (and in fact on all n-tori which appear after further bifurcations) is the constant velocity flow. This is a much more stringent requirement than the one of the Ruelle-Takens scenario. While the latter is fulfilled on an open set of vector fields, the former does not hold on any open set of vector fields and is not even generic, i.e., it does not hold on any countable intersection of dense open sets (called a residual set). But genericity is perhaps a minimal way of saying that something is likely, and thus the Landau scenario is not likely. (In particular, if two properties are generic, they hold simultaneously on a residual set, and residual sets are more or less the weakest possibility for this simultaneity property to hold.)

Returning to the Ruelle-Takens scenario, we add a word of caution. While it is true that the set of vector fields with strange attractor is open near the constant vector fields, this does not mean that this set is large in the measure theoretic sense. We can visualize the situation in the space of vector fields near the constant vector fields as in Fig. 6.



. 6. Measure theoretic situation for the Ruelle-Takens-/house scenario.

Experimental evidence and its measurement

In order to describe how the appearance of the scenario manifests itself in measurements and to show the measurable consequences of the presence of strange attractor, let us reformulate the scenario: *If a system undergoes three Hopf bifurcations, starting from a stationary solution, as a parameter is varied, then it is likely that the system possesses a strange attractor with sensitivity to initial conditions after the third bifurcation.*

The *power spectrum* of such a system will exhibit one, then two, and possibly three independent basic frequencies. When the third frequency is about to appear, simultaneously some broad-band noise will appear if there is a strange attractor. This we interpret as chaotic, turbulent evolution of the system. Experiments have been performed on the formation of Taylor vortices between rotating cylinders and the Rayleigh-Bénard convection (see Figs. 7 and 8; for a review, from which these figures are taken, see Swinney and Gollub, 1978). They can be interpreted in the sense of the Ruelle-Takens-Newhouse scenario. It should also be stressed that measurements of time correla-

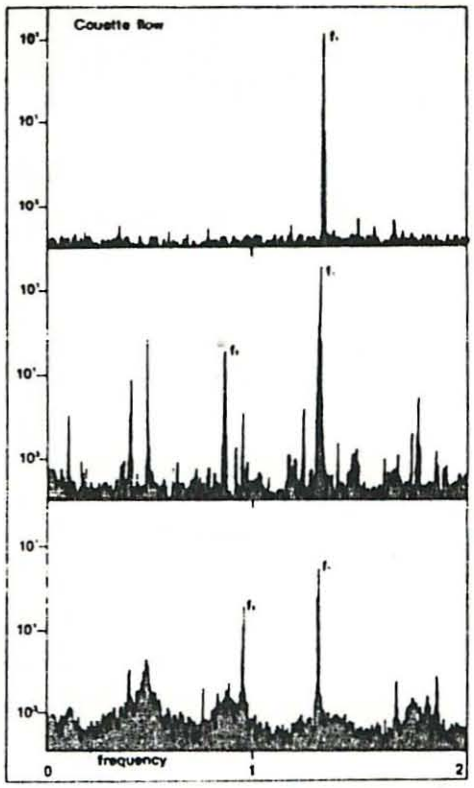


FIG. 7. Power spectrum of velocity in rotating cylinders driven at three different speeds.

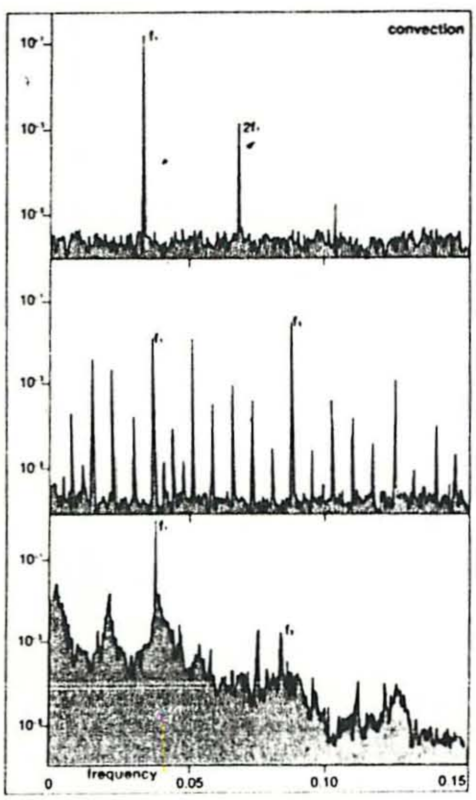


FIG. 8. Power spectrum of heat transport at different heating in Rayleigh-Bénard convection.

tions (measures of k-tuples $x_l, x_{l+1}, \ldots, x_{l+k-1}$ as a function of l) are very useful indicators about flows in general (Takens, 1980; Roux *et al.*, 1980), and allow one in some sense to reconstruct the dynamical system.

E. The influence of noise

The Ruelle-Takens scenario is *not* destroyed by the addition of small external noise to the evolution equations. This result, which is somewhat counterintuitive, will be explained in more detail in the final section. In effect, the chaos of the scenario is so strong that order cannot be accidentally established by small noise terms, much like a very attracting fixed point is locally not much altered by noise, and globally there is at most a small probability to change stochastically from one basin of attraction to another (Kifer, 1974; Ventsel and Freidlin, 1970).

V. THE FEIGENBAUM SCENÁRIO

A. Description

We start with the description of a géneral framework. Assume we are in the presence of a one-parameter family of vector fields v_μ in R^m (we conjecture that the results extend to the case $m=\infty$), where μ is the parameter. Assume each v_μ has a periodic orbit, and assume there is a piece of hyperplane of dimension $m-1$, transversal to this periodic orbit, for which the Poincaré map P_μ can be defined (Fig. 9). The scenario will make predictions about these Poincaré maps and hence for the corresponding flow.[9]

Now fix m. Two objects, Φ_m and W_m, whose existence is asserted by a mathematical theory, will be of fundamental importance in describing the scenario, namely, there is a neighborhood D_m of $[0,1]\times\{0\}^{m-2}$ in C^{m-1} and on this neighborhood an analytic function Φ_m: $C^{m-1}\to C^{m-1}$ whose restriction to R^{m-1} is real. In the space of analytic functions on D_m (with, for example, the sup norm) there is an open disk W_m of codimension one, containing Φ_m. The existence of the two objects Φ_m and W_m is assured through an extension of Feigenbaum's original theory (Feigenbaum, 1978, 1979a) ($m=2$, one-dimensional maps) by Collet, Eckmann, and Lanford (1980) and Collet, Eckmann, and Koch (1981).

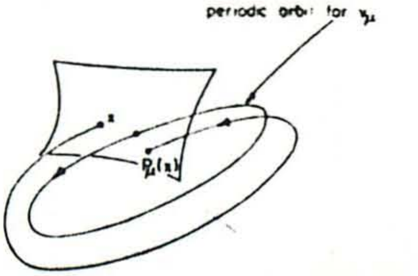


FIG. 9. Phase portrait illustrating Poincaré section of v_μ.

[9]These ideas were first explained in Eckmann (1980). See also Collet and Eckmann (1980).

B. Assumptions

The scenario assumes that P_μ extends to an analytic function on D_m and that the curve $\mu\to P_\mu$ transversally crosses W_m near Φ_m.

Under these hypotheses one can assert

(1) The family P_μ has an infinite sequence of period doubling bifurcations of stable periodic orbits at parameter values μ_1 (period 1-2), μ_2 (period 2-4), ..., μ_{j+1} (period 2^j-2^{j+1}) (the sequence might only start at some high j).

(2) $\lim_{j\to\infty}\mu_j=\mu_\infty$ exists.

(3) At $\mu=\mu_\infty$, P_μ has an aperiodic attractor (a stable periodic orbit of "period 2^∞"). The action on the attractor is ergodic, but not mixing (in particular, there is no sensitive dependence on initial conditions).

(4) There is a universal number $\delta=4.66920\ldots$ such that

$$\lim_{j\to\infty}\frac{1}{j}\log|\mu_j-\mu_\infty|=-\log\delta\,.$$

One even has

$$|\mu_j-\mu_\infty|\sim \mathrm{const}\,\delta^{-j} \text{ as } j\to\infty\,.$$

C. Remarks

(1) The bifurcations of the orbit structure of P_μ are *pitchfork bifurcations*, i.e., a stable fixed point loses its stability and gives rise to a stable periodic orbit as the parameter is changed. This corresponds to a crossing of one eigenvalue of the tangent map DP_μ through -1 (Fig. 10).

(2) One can show that any suitable property (such as bifurcation) which can be described by a coordinate independent codimension 1 surface in the space of functions on D_m will double its spatial structure in phase space in the same way as the periodic orbits, i.e., it will split in 2, 4, 8, ... pieces. Typically, such surfaces are given by a single functional relation, e.g., fixing the value of a derivative at a fixed point.

(3) A similar scenario exists for area-preserving (=Hamiltonian) maps of the plane to itself, but with 8.721... as the universal constant instead of $\delta=4.66920\ldots$ (Collet, Eckmann, and Koch, 1980; Greene *et al.*, 1981).

(4) The scenario can be somewhat extended under the assumption of very strong friction. This has the effect of making the situation very similar to the case of maps of the interval to itself. Then one can show that if the system has transitions from periods 1 to 2 and 2 to 4 at values μ_1 and μ_2, respectively, a stable period 3 with a large basin of attraction near

$$\mu=\frac{(\delta\mu_2-\mu_1)}{(\delta-1)}-\frac{\delta(\mu_1-\mu_2)0.803}{(\delta-1)}$$

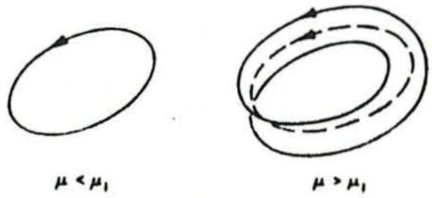


FIG. 10. Example of a pitchfork bifurcation for a flow.

e expected.

After the cascade of period doublings, one expects ond the accumulation point μ_∞ an *inverse* cascade of periods.

The physical interpretation of the Feigenbaum scenario can be brought to a more appealing form than for the Ruelle-Takens scenario, because the statement deals with *all* curves which cross W_∞ transversally. On the other hand, it is only a statement about a very small parameter range, and point (B.4) describes nothing more than a critical index.

D. Interpretation

In an experiment, if one observes subharmonic bifurcations at μ_1, μ_2, then, according to the scenario, it is very probable for a further bifurcation to occur near $\mu_3 = \mu_2 - (\mu_1 - \mu_2)/\delta$, where $\delta = 4.66920\ldots$. In addition, if one has seen three bifurcations, a fourth bifurcation becomes more probable than a third after only two, etc. At the accumulation point, one will observe aperiodic behavior, but no broad-band spectrum.

E. Experimental evidence

This scenario is extremely well tested on numerical and physical grounds. The period doublings have by now been observed in most current low dimensional dynamical systems (Hénon map, Lorenz equations, forced oscillator with friction, etc). Experiments with liquid helium have confirmed the predictions.

F. Measurement

In all numerical examples, the bifurcations are found by a direct analysis of the orbits and of their stability. The experiments on liquid helium produce power spectra. Feigenbaum has given a nice prediction of how the power spectrum evolves as a function of the parameter (see Fig. 11). At each successive bifurcation a new frequency is born. The mean of the squares of the new amplitudes is then expected to rise until it stops about 13.5 db below the level of its predecessors (Feigenbaum, 1979b, 1980; Nauenberg and Rudnick, 1981; Collet, Eckmann, and Thomas, 1981).

The measured power spectrum of Libchaber and Maurer (1980)[10] for the heat transport by convection of liquid helium, heated from below, shows a sequence of period doubling bifurcations. The power goes down by about 10 db per doubling, but the apparent discrepancy with the prediction of the scenario may be ascribed to not yet having reached the asymptotic regime (Fig. 12). The prediction (5) above has recently been seen by Libchaber (1981) [Fig. 12(c)].

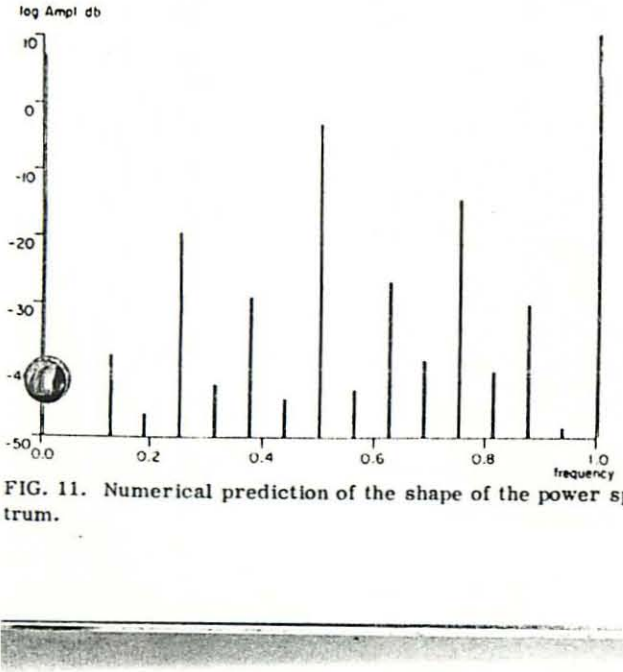


FIG. 11. Numerical prediction of the shape of the power spectrum.

G. The influence of noise (Crutchfield *et al.*, 1980)

Again we postpone a detailed description of the influence of noise. Since the structure of the periodic orbit must acquire finer and finer length scales as the parameter approaches μ_∞, it is clear that even very small noise will eventually play a role. There exist estimates on the relation between the noise level and the maximal period which can be observed. This is of course related to the power spectrum described above.

VI. THE POMEAU-MANNEVILLE SCENARIO

A. Description

This scenario (Pomeau and Manneville, 1980; Manneville and Pomeau, 1980) has been—correctly—termed *transition to turbulence through intermittency*. Its mathematical status is somewhat less satisfactory than that of the two other scenarios presented here. This is because the parameter region the scenario intends to describe contains an infinity of (very long) stable periods, and because there is no mention as to when the "turbulent" regime is reached or what the exact nature of this turbulence is. We nevertheless examine it here because of its esthetic and conceptual beauty.

While the two other scenarios have been associated with Hopf bifurcations (Ruelle-Takens) and pitchfork bifurcations (Feigenbaum), this one is associated with a "saddle node bifurcation," i.e., the collision of a stable and an unstable fixed point which then both disappear (into complex fixed points).

The general idea is best explained for the simple example of a one-parameter family of iterated maps on the unit interval, $x_{n+1} = f_\mu(x_n)$. We take $f_\mu(x) = 1 - \mu x^2$, which for $\mu \in [0,2]$ maps $[-1,1]$ into itself. The function $f_\mu^3 = f_\mu \circ f_\mu \circ f_\mu$ can be shown to have a saddle node for $\mu = \frac{7}{4}$. For $\mu > 1.75$, f_μ^3 has a stable periodic orbit of period three, and an unstable one nearby. The two collide at $\mu = 1.75$, and both have then eigenvalue 1. See Fig. 13.

For μ slightly below 1.75, the local picture near $x = 0$ is shown in Fig. 14. It can be shown that if $\mu - 1.75 = O(\varepsilon)$ then a typical orbit will need $O(\varepsilon^{-1/2})$ iterations to cross a fixed small x interval around $x \sim 0$. As long as the orbit is in this small interval, an observer will have the impression of seeing a periodic orbit of period three. Once one has left the small interval, the iterations of the map will look rather like those of a chaotic map [a consequence of a

[10] See Collet and Eckmann (1980), pp. 39 and 42 for a list of tests. In particular, beautiful experiments on liquid helium were performed by Libchaber and Maurer (1980).

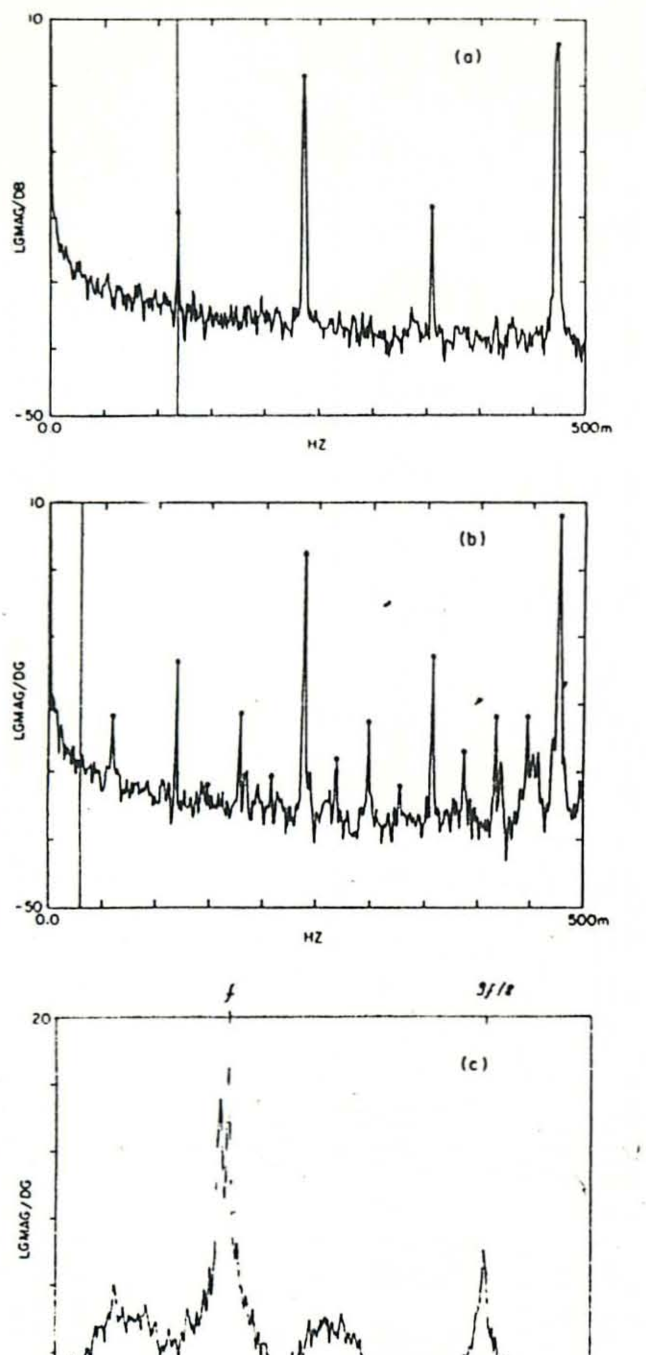


FIG. 12. Power spectra for two values of heating. (c) Observation of the noisy period 8.

result of Misiurewicz; see Collet and Eckmann (1980), Theorem 5.2.2]. Thus this map can be called intermittently turbulent (see Fig. 15).

The problem with this argument comes in the splitting into two regions. It is true that the iterated map may have sensitivity to initial conditions for $x \in$ small intervals around contact points. But this destabilizing effect may be lost whenever one passes near the contact point. In fact, we conjecture that this will happen for an infinity of parameter values near to, and just below $\mu = 1.75$. For these parameter values, one will have (very long) stable periods, but no chaos. On the

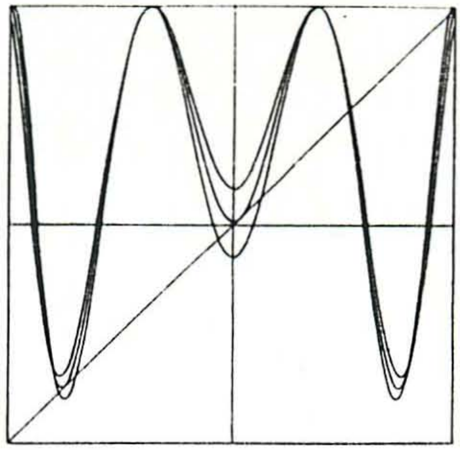


FIG. 13. Graph of f_μ^3 for three values of μ.

other hand, we also conjecture that a modification of the proof of Jakobson (1980) would show that truly aperiodic behavior with sensitivity to initial conditions occurs for a set of parameter values of positive Lebesque measure near 1.75.

B. Assumptions

We can now formulate a reasonable version of this scenario for general dynamical systems.

Assume a one-parameter family of dynamical systems has Poincare maps close to a one-parameter family of maps of the interval, and that these maps have a stable and unstable fixed point which collide as the parameter is varied. Then, as the parameter is varied further to μ from the critical parameter value μ_c, one will see intermittently turbulent behavior of random duration, with laminar phases of mean duration $\sim(|\mu - \mu_c|^{-1/2})$ in between.

C. Interpretation

The difficulty with this scenario is that it does not have any clear-cut precursors, because the unstable fixed point which is going to collide with the stable fixed point (respectively periodic orbit) may not be visible. One can think of two ways out of this problem. The first would be that increasingly long transients can be observed before the two fixed points (periodic orbits) collide. The second kind of precursor is a cascade of *inverse* pitchfork bifurcations, and, at the "end" of this, the intermittent transition to turbulence (Collet and Eckmann, 1980).

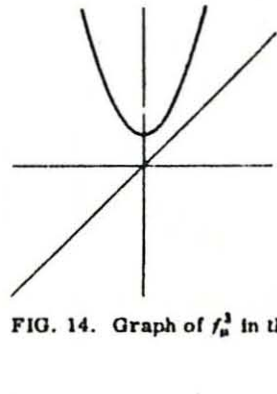


FIG. 14. Graph of f_μ^3 in the vicinity of the origin.

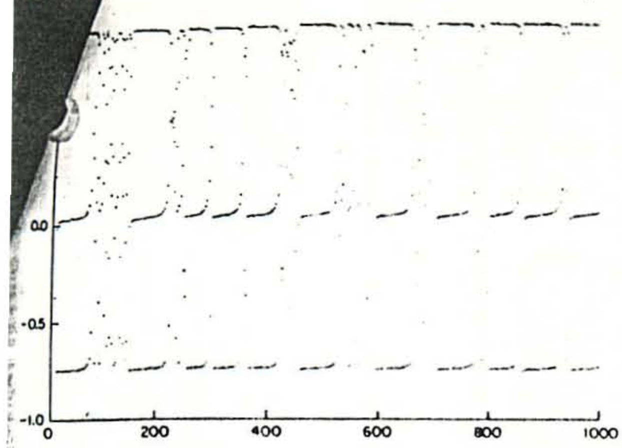


FIG. 15. Graph describing $f_\mu^n(0)$ as a function of n in the neighborhood of $\mu \simeq 1.75$, and indicating the existence of an intermittent turbulence.

D. Experimental evidence

Pomeau and Manneville based their work on observations for the Lorenz system. Intermittent transitions to turbulence can be seen in many physical experiments. The only ones which seem to agree with the scenario described above are those of Maurer and Libchaber (1980), Bergé *et al.* (1980), and Pomeau *et al.* (1981). They exhibit intermittent transition to aperiodic behavior, but more work needs to be done to show that these are really instances of the scenario described al .

E. Measurement

We have already discussed the difficulties of detecting the scenario. We add here only that one should not look at power spectra in this case, but rather at real-time measurements.

F. The influence of noise

As the parameter value at which the two fixed points collide is a critical point, the influence of noise is relevant. This has been first exhibited by Mayer-Kress and Haken (1981). A more detailed analysis of the tunneling through the region of contact shows that certain scaling relations hold between the noise level and the distance from the critical parameter value (Eckmann *et al.*, 1981).

VII. THE INFLUENCE OF EXTERNAL NOISE ON SCENARIOS

It seems to be a widespread opinion that external noise is relevant

(a) for the appearance of (even weak) turbulence and chaotic behavior and

(b) for the form, amplitude, and spectrum of the turbulence, once it has appeared.

The foregoing discussion of attractors and of the scenarios should have shown that this opinion is wrong for case (a)—*ergodic behavior is possible, and quite common, for dynamical equations without external noise*. In this section, we shall examine case (b) and see that the nature of chaotic systems may be totally insensitive to small external noise. The systems most sensitive to noise seem to be deterministic systems near transition (bifurcation) points.

This insensitivity to noise is surprising and at first sight counterintuitive. It has been discovered by Kifer (1974), whose work is an extension of a paper by Ventsel and Freidlin (1970). Kifer's theorem states that for a dynamical system with an Axiom A attractor, which has an invariant measure ν, the following is true: Given any reasonable small noise, going to zero with σ, consider the corresponding invariant measure ν_σ. [Under suitable assumptions, the measures ν and ν_σ are given, for discrete mappings f as follows:

$$\int d\nu(x)h(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} h(f^k(y))$$

for Lebesque-almost every y, and every continuous h.] The density of the measure ν_σ, given a noise with transition probability $\rho_\sigma(x, y)$ [and an iteration scheme $x_{n+1} = f(x_n) + \xi_\sigma(x_n)$, where ξ_σ is a random variable with density $\rho_\sigma(x_n, .)$] satisfies

$$\nu_\sigma(x) = \int \rho_\sigma(f(y), x - f(y))\nu_\sigma(y)dy\,.$$

Theorem (Kifer, 1974). *ν_σ converges weakly to ν as $\sigma \to 0$ (i.e., all expectation values of bounded observables converge).*

This tells us, then, that if the noise is sufficiently small, the corresponding probability distributions (ν and ν_σ) are as close to each other in the weak-* topology as we wish. This result is astonishing, because any nontrivial (strange) Axiom A attractor is full of hyperbolic points, and one could think that a small random deviation might get amplified away from any deterministic path. But the celebrated "shadowing lemma" leads to a different conclusion. With high probability, the sample paths of the problem with external noise follow *some* orbit of the deterministic problem arbitrarily closely. This bounds ν_σ by ν (up to small errors). On the other hand, the central limit theorem shows that ν is bounded by ν_σ: For every deterministic orbit, there are many sample paths which follow it rather closely.

We next discuss the influence of noise on the Feigenbaum scenario. It is known (Collet, Eckmann, and Lanford, 1980; Collet *et al.*, 1981; Feigenbaum, 1978, 1979a) that the smallest scales of the period 2^n are of approximate size $O(\lambda^{2n})$, with $\lambda = .3995..$ (another universal constant). Thus it is obvious that even small noise can wipe out the finest structures of the orbit, and hence the orbit itself, provided n is sufficiently large. The question then is how large the noise may be if we want to see a period 2^n. Crutchfield *et al.*, (1980) give a heuristic argument with the following conclusion. Denote, for each k, by ξ_k the independent random variables with mean zero and density ρ. Let f_μ be a one-parameter family of maps of the interval, with μ so chosen that the accumulation of period doublings is at $\mu = \mu_\infty = 0$. Consider the stochastic iteration equation

$$x_{k+1} = f_\mu(x_k) + \xi_k\,.$$

and define $\nu_{\mu,\sigma}$ the corresponding invariant density. Then one has the approximate identity

$$\lambda \nu_{\mu\delta,\kappa\cdot\rho\star\kappa}(\lambda x) \sim \nu_{\mu,\rho}(x),$$

with $\lambda = 0.39953\ldots$, $\delta = 4.66920\ldots$, $\kappa = 6.619\ldots$. In words, in order to see twice the period, the noise must have a variance about κ times smaller. [Note that this is very close to the ratio of the amplitudes between a frequency and its subharmonic, which has been estimated by Feigenbaum (1979b) to be about 6.60... .]

In the Pomeau-Manneville scenario, the influence of noise can be modeled as follows (Eckmann *et al.*, 1981). In the "laminar" region, i.e., when the iteration steps are small, one can model the iteration scheme

$$x_{n+1} = x_n + x_n^2 + \varepsilon + \sigma\xi_n,$$

with ξ_n independent stochastic variables, by the stochastic differential equation

$$dx = (x^2 + \varepsilon')dt + \sigma' d\omega,$$

where ω is white noise, and $\varepsilon' = \varepsilon$, $\sigma' = \sigma \operatorname{Exp}(\xi^2)^{1/2}$. The estimated time to cross the laminar region is then easily seen to be a stopping time for the differential equation, and an analysis of its solution shows that the fraction of time spent in the laminar region scales approximately as $\varepsilon^{-1/2}T(\sigma'/\varepsilon^{3/4})$, where T is a universal function.

See Table II for a summary of these three scenarios.

ACKNOWLEDGMENT

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$$\frac{du}{(1+u^2)^{1/3}} \cdot \frac{-2}{3}\frac{1}{u^{5/3}}\,du$$

$u = \sinh\theta.$

$$\frac{\cosh\theta}{\cosh^{2/3}\theta}\;\frac{1}{\cosh^{5/3}\theta}.$$

$$\int \delta(x^2-a^2)\,dx$$

$x^2-a^2=u$

$du = 2x\,dx$

$= 2\sqrt{u^2+a^2}\,dx.$

$$\int \delta(u)\frac{du}{2\sqrt{u^2+a^2}} = \int \delta(u)\,du$$

$\Rightarrow \pm\frac{1}{2a}\,\delta(u)$

$x=a \quad x=-a.$

$$\frac{1}{2a}\left(\delta(x+a)-\delta(x-a)\right)$$

$$\frac{dx}{(1+\cancel{x}^3)^{1/3}} \quad -\frac{1}{x^2}\,dx.$$

$\frac{1}{x^3} = u^2$

$du = -\frac{1}{x^2}\,dx$

$\frac{-3}{x^4}\,dx = 2u\,du.$ $= \frac{-2}{3}u^{-5/3}\,du.$

$$dx = \frac{-2}{3}\,\frac{u x^4\,du}{1+\frac{1}{x^3}} = u.$$

$du = -3dx\,\frac{1}{x^4}.$

$$\frac{dx}{\left(\frac{\cancel{1}}{\cancel{1}} + \frac{1}{x^3}\right)^{1/3}}.$$

$$\frac{-1}{3}\frac{du}{(u-1)^{4/3}}\frac{1}{u^{1/3}}.$$

$u = \frac{1}{x}.$

$du = -\frac{dx}{x^2} =$

$$\frac{1}{(u-1)}\;\frac{1}{\left((u-1)u\right)^{\frac{1}{3}}}$$