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Some Characterizations of Strange Sets

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A thermodynamic formalism is exhibited that is the canonical version of Halsey *et al.*'s microcanonical formulation. This formalism is applied to a four-scale Cantor set and it is shown that the singularity spectrum fails to uniquely encode the underlying dynamics.

KEY WORDS: Spectrum of singularities; thermodynamic formalism; transfer matrix; scaling function; dynamical systems.

1. INTRODUCTION

Halsey *et al.*[1] have introduced a method of extracting a spectrum of scalings from experimentally or numerically generated strange sets. We shall show that this method is the microcanonical version of the canonical thermodynamic formalism of Ruelle *et al.*[2] Hereafter we shall refer to Halsey *et al.* as MP (microcanonical paradigm), and to Ruelle *et al.* as CP (canonical paradigm).

Our presentation consists in constructing the canonical ensemble in a form suitable for theoretical calculations on a dynamical system, relating it to the microcanonical formulation, and producing a conversion dictionary between the standard CP vocabulary and the MP functions as defined in Ref. 1. We then demonstrate that in the simplest nontrivial example, the Cantor set characterized by four scales, MP depends only upon three independent combinations of scales, and thus fails to uniquely characterize the set.

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2. THE SINGULARITY SPECTRUM ACCORDING TO CP

Consider a dynamical system whose attractor can be hierarchically presented as a set of N_n intervals $I_k^{(n)}$, $k = 1,..., N_n$, of lengths $\Delta_k^{(n)}$ at the nth level.

Recipe. Construct the canonical "free energy" according to the definition

$$N_n^{-F(\beta)} = \sum_k |\Delta_k^{(n)}|^\beta \tag{2.1}$$

where we understand that asymptotically in n, F becomes independent of n. The relation of the MP functions defined by Halsey *et al.* to the CP quantities is given by the following dictionary:

$$\alpha = 1/F'(\beta) \tag{2.2}$$

$$f = \beta - F(\beta)/F'(\beta) \tag{2.3}$$

$$q = -F(\beta) \tag{2.4}$$

$$\tau = -\beta \tag{2.5}$$

$$D_q = \beta/[1 + F(\beta)] \tag{2.6}$$

By (2.2) and (2.3) we mean that $f(\alpha)$ implicitly parametrized by β *is* $f(\alpha)$ of MP. By (2.4) and (2.5), a plot of q versus $\tau(q)$ is precisely $F(\beta)$. Since q versus τ *is* what is experimentally available, and F is the natural theoretical object, one could probably dispense with f and use (2.2) to talk about the "range of scales."

Observe in (2.1) that, since $|\Delta| < 1$, $F(\beta)$ is a monotonic increasing function diverging with β at $\pm\infty$. It has a unique zero, which is an upper bound to, or the Hausdorff dimension itself. By (2.4) and (2.6) this is D_0. By (2.2) and (2.6), $D_{\pm\infty} = \alpha_{\pm\infty}$.

We shall show that (2.2) and (2.3) are the quantities defined in MP, while (2.4)–(2.6) are internal consequences of MP. However, first we shall explore CP with two examples.

3. EXAMPLES

3.1. Two-Scale Cantor Set

Consider a Cantor set whose nth level consists of intervals

$$\Delta^{(n)}(\varepsilon_n,..., \varepsilon_1) = \sigma_1^n(\sigma_2/\sigma_1)^{\Sigma_1^n \varepsilon}, \qquad \varepsilon_i = 0, 1 \tag{3.1}$$

$$N_n = 2^n \tag{3.2}$$

Observe that

$$\frac{\Delta^{(n)}(\varepsilon_n, \varepsilon_{n-1},..., \varepsilon_1)}{\Delta^{(n-1)}(\varepsilon_{n-1},..., \varepsilon_1)} = \sigma(\varepsilon_n) = \sigma_1^{1-\varepsilon_n}\sigma_2^{\varepsilon_n} \tag{3.3}$$

Denoting the number of iterates of the critical point by $t = \varepsilon_1 + \cdots + 2^{n-1}\varepsilon_n$, we have that $\varepsilon_n = 0$ denotes the first quarter of t through N_{n+1}, and $\varepsilon_n = 1$ the second. Thus, σ_1 and σ_2 represent the leading two-scale approximation to the scaling function of Ref. 3, hereafter referred to as SP (the scaling paradigm).

By (2.1),

$$\begin{aligned} e^{-nF(\beta)\ln 2} &= \sigma_1^{\beta n} \sum_{\{\varepsilon_1,..., \varepsilon_n\}} (\sigma_2/\sigma_1)^{\beta\Sigma_1^n \varepsilon} \\ &= \sigma_1^{\beta n}\left[\sum_{\varepsilon=0}^{1} (\sigma_2/\sigma_1)^{\beta\varepsilon}\right]^n \\ &= (\sigma_1^\beta + \sigma_2^\beta)^n \end{aligned}$$

or

$$F(\beta) = -\ln(\sigma_1^\beta + \sigma_2^\beta)/\ln 2 \tag{3.4}$$

Equation (3.4) together with (2.2) and (2.3) reproduces the graph of $f(\alpha)$ of MP.

3.2. Four Scale Contor Set

Consider a Cantor set whose nth level consists of intervals characterized by four scales

$$\begin{aligned} &\Delta^{(n)}(\varepsilon_n,..., \varepsilon_1) \\ &\quad = \sigma_1^n(\sigma_2/\sigma_1)^{\Sigma_1^n \varepsilon}(\sigma_3/\sigma_1)^{\Sigma_1^{n-1} \varepsilon}(\sigma_1\sigma_4/\sigma_2\sigma_3)^{\Sigma_1^{n-1} \varepsilon_i \varepsilon_{i+1}}, \qquad \varepsilon_i = 0, 1 \end{aligned} \tag{3.5}$$

As in (3.3),

$$(\Delta^{(n)}/\Delta^{(n-1)})^\beta = \sigma^\beta(\varepsilon_n, \varepsilon_{n-1}) \equiv \sigma^\beta(\varepsilon, \varepsilon') = T_{\varepsilon'\varepsilon}$$

where

$$T = \begin{pmatrix} \sigma_1^\beta & \sigma_2^\beta \\ \sigma_3^\beta & \sigma_4^\beta \end{pmatrix} \tag{3.6}$$

$\sigma_1,..., \sigma_4$ is just the leading four-scale approximation to the SP scaling function $\sigma(t)$, with σ taken as constant at σ_i on intervals of 1/8 of N_{n+1}.

We see by (3.6) that T, i.e., the SP scaling function, is simply the transfer matrix of the Ising model that (3.5) produces in canonical ensemble when substituted in (2.1).

Denoting the larger eigenvalue of T by $\lambda(\beta)$, it follows from (2.1) that

$$F(\beta) = -\ln \lambda(\beta)/\ln 2 \tag{3.7}$$

where

$$\lambda(\beta) = \frac{\sigma_1^\beta + \sigma_4^\beta}{2} + \left[\left(\frac{\sigma_1^\beta - \sigma_4^\beta}{2}\right)^2 + (\sigma_2\sigma_3)^\beta\right]^{1/2} \tag{3.8}$$

Thus, the theoretical $f(\alpha)$, D_q or whatever, is now available for this four-scale Cantor set. The reader *must* realize that period doubling dynamics *does* produce four measurable scales. The thermodynamic quanity F, however, depends on σ_2 and σ_3 only through the *combination* $\sigma_2\sigma_3$.

That is, the MP description $f(\alpha)$ of scalings of strange sets other than the trivial two-scale Cantor set is *infinitely degenerate* over internal scales: already in the four-scale approximation to the period doubling attractor, MP fails to distinguish the period doubling attractor from the one-dimensional family of other strange sets with the same product $\sigma_2\sigma_3$. None of this massaging of experimental data can justify the claim that a definite dynamics qua metric has been observed.

4. THE CONNECTION OF CP TO MP

Starting with (2.1), write

$$N_n^{-F(\beta)} = \sum_\mu N^{-\beta\mu} \left(\sum_{h(m_1,\ldots,m_l)=\mu} \right) \tag{4.1}$$

where we have defined

$$-\ln \Delta_k^{(n)}/\ln N_n \underset{n\to\infty}{\sim} h(m_1,\ldots,m_l) \tag{4.2}$$

with $m_1,\ldots,m_l$ a sufficient set of *intensive* variables that label the kth interval $I_k^{(n)}$. We expect in (4.2) that $\Delta_k^{(n)}$ and N_n have exponential dependence on n. The relation (4.2) defines h to be the Hamiltonian per "site" n. In the example (3.5) we see that h is an Ising Hamiltonian, and requires two intensive parameters

$$\frac{1}{n}\sum_1^n \varepsilon_i \qquad \text{and} \qquad \frac{1}{n}\sum_1^{n-1} \varepsilon_i\varepsilon_{i+1}$$

to specify a definite value of h, and so Δ.

Consider the microcanonical ensemble in (4.1):

$$\left(\sum_{h(m)=\mu} \right) \sim N_n^{s(\mu)} \tag{4.3}$$

defining the microcanonical entropy $s(\mu)$. Thus, (4.1) becomes

$$N_n^{-F(\beta)} = \sum_\mu N^{s(\mu)-\beta\mu} \tag{4.4}$$

The microcanonical version of F is then

$$F = \beta\bar{\mu} - s(\bar{\mu}), \qquad \text{where} \quad \beta = s'(\bar{\mu}) \tag{4.5}$$

determines the maximizing $\bar{\mu}(\beta)$. Properties (2.2) and (2.3) now follow from the identifications

$$\alpha = 1/\bar{\mu}, \qquad f = s(\bar{\mu})/\bar{\mu} \tag{4.6}$$

and Legendre transformation.

It is now easy to see that α and f of (4.6) are just those of MP. At $h=\bar{\mu}$, the length of an interval by (4.2) is $l = N_n^{-\bar{\mu}}$. Since each piece is visited with probability N_n^{-1}, this is $p(l) = l^{1/\bar{\mu}} \equiv l^\alpha$. The microcanonical sum in (4.3) is the number of intervals with $\alpha = 1/\bar{\mu}$, which in MP is $l^{-f(\alpha)}$. This verifies the second part of (4.6). Thus, MP = CP.

5. CONCLUSION

In practice, MP is a convenient tool for extracting three numbers characterizing a strange set: the Hausdorff dimension, the minimal scaling, and the maximal scaling. While we, too, believe, along with Halsey *et al.*, that MP represents a significant breakthrough in numerical analysis of strange sets, MP encodes only a small part of the metrical structure of the underlying dynamics.

The standard thermodynamic CP formalism is the correct machine for the theoretical detrmination of MP quantities; formula (3.8) would have been very hard to come by if computed in MP. CP makes it very clear that a *full* metric invariant, such as the SP scaling function $\sigma(t)$, is the necessary theoretical ingredient for CP to transform into MP. Whether or not σ is ugly and perverse turns on one's viewpoint. Since σ is defined only on the Cantor set—which is all that we are describing—it is perhaps worth noting that σ here is not only continuous, but quite differentiable on the strange set S.

In this connection Ref. 4 might also be of interest.

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Scaling Spectra and Return Times of Dynamical Systems

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The grand canonical version of the spectrum of singularities formalism is presented, relying naturally upon certain Markov transition graphs. The structure of a graph is simply determined by the close return times of the dynamical system described. Thus, an intimate connection exists between the shape of the singularity curve and a small but interesting set of dynamical properties.

KEY WORDS: Spectrum of singularities; Markov graphs; return times; scaling function; dynamical systems; thermodynamic formalism.

1. INTRODUCTION

In a previous note,[1] hereafter referred to as CP, we realized that the formalism of Halsey *et al.*[2] is a microcanonical version of a canonical formalism intimately related to Hausdorff measure. In CP we discovered that the scaling function of Ref. 3 for period doubling serves as the transfer matrix for a 1D Ising model. The number of sites n in CP is the level of recursive construction of the attractor, so that one takes the thermodynamic limit $n \to \infty$ to exactly recover the attractor. Since computations performed at finite n can have constraints on relevant combinatorics, it is natural to follow tradition and construct a grand canonical formulation. We do so, and immediately realize that the grand sum is obtained by summing all paths on a graph with directed links—that is, on a Markov transition graph. It is easiest to fix the ideas through example. We choose golden mean rotation for this purpose.

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Let me summarize what will be done. The exponential of the free energy at the nth level of construction is a sum of interval lengths raised to power β. Each such length is the product of n appropriate scaling factors. This sum is multiplied by z^n, and n summed over. This means each scaling factor is to be multipled by z, and arbitrary products of successive scalings formed. In successively more exact approximations there are a finite number of distinct scalings which can follow one another by well-defined rules. Thus, we have a graph whose nodes are to be correctly linked by directed links, each link having weight z times a scaling (a definite number) to the β power. All paths through the graph are to be formed and summed, thereby producing one over a characteristic determinant which must vanish for $n \to \infty$. This occurs for the zero $z(\beta)$, which is simply the inverse of the leading eigenvalue of the transfer matrix, and so provides the free energy. Elementary circuit manipulations make calculations trivial to perform.

Through example, we will observe that the number of nodes and allowed links on a graph are determined by the structure of close return times. The most prominent properties of the free energy (or f versus α curve; for example, $\alpha_{\min}$ and $\alpha_{\max}$ of Ref. 2) turn out to depend upon the lowest order cycles on the graph. Thus, a deep connection is seen to exist between the nature of close return times and the ensuing f versus α curve. That is, with no more theoretical information than the form of close returns, phenomonologically correct f versus α curves are determined.

2. GRAND CANONICAL FORMALISM

From CP,

$$N_n^{-F_n(\beta)} = \sum |\Delta_t^{(n)}|^\beta, \qquad \text{where} \quad F_n(\beta) \underset{n\to\infty}{\simeq} F(\beta) \tag{1}$$

Defining

$$e^{-G(z,\beta)} \equiv \sum_n z^n N_n^{-F_n(\beta)} \tag{2}$$

we obtain the canonical value of G as the summand, for $n = \bar{n}$, that is stationary in n. That is,

$$G = -\bar{n} \ln z + F_n \ln N_{\bar{n}} \tag{3}$$

with $0 = \partial G/\partial \bar{n}$ determining $\bar{n}(z)$. Thus,

$$\partial G/\partial z = -\bar{n}/z \tag{4}$$

For $\bar{n} \to \infty$, write

$$G = \ln u \tag{5}$$

so that (4) becomes

$$u'/u = -\bar{n}/z$$

and $u \to 0$ as $\bar{n} \to \infty$, i.e.,

$$u(\beta, z) = 0 \tag{6}$$

With $N_n \sim a^n$, (3) becomes

$$\ln a\, F(\beta) = \ln z + \lim_{n\to\infty} G/\bar{n} = \ln z + \lim_{u\to 0}\left(-\frac{1}{zu'} u \ln u\right) = \ln z$$

Together with (6), we thus have the recipe

$$F(\beta) = \ln z(\beta)/\ln a \tag{7}$$

where

$$u(\beta, z(\beta)) \equiv 0 \tag{8}$$

Next, substitute (1) in (2):

$$\frac{1}{u} = \sum_n z^n \sum_t |\Delta_t^{(n)}|^\beta$$

$$= \sum_n z^n \sum_{\{\varepsilon_1,\ldots,\varepsilon_n\}} |\Delta^{(n)}(\varepsilon_n,\ldots,\varepsilon_1)|^\beta \tag{9}$$

where a logarithmic basis $\varepsilon_1,\ldots,\varepsilon_n$ labels the index t of a particular nth level interval. For the case of a dynamical system, t is simply the number of time steps required to image some one $\Delta_0^{(n)}$ into $\Delta_t^{(n)}$, and we write

$$t = \varepsilon_1 T_1 + \cdots + \varepsilon_n T_n, \qquad T_1 < \cdots < T_n \tag{10}$$

where the T_i are successively longer close return times.

Let us write

$$\frac{\Delta^{(n)}(\varepsilon_n, \varepsilon_{n-1},\ldots,\varepsilon_1)}{\Delta^{(n-1)}(\varepsilon_{n-1},\ldots,\varepsilon_1)} \equiv \sigma_n(\varepsilon_n,\ldots,\varepsilon_1) \tag{11}$$

The point of Ref. 3 is that the scalings σ_n depend successively (exponentially) more weakly on the lower ε's, and become independent of n asymptotically. Just how many of $\varepsilon_n, \varepsilon_{n-1},\ldots$ are to be kept determines successive approximations. By Refs. 3 and 4 for period doubling and Ref. 5 for golden mean rotation, very few of the ε's determine excellent approximations.

Equation (11) then says that $\Delta^{(n)}$ is a product of n σ's, with each successive σ depending on all but the leftmost ε of its predecessor. The σ's thus are transition amplitudes between one set of $\varepsilon_n,\ldots,\varepsilon_{n-r}$ in the rth approximation and all those $\varepsilon_{n-1},\ldots,\varepsilon_{n-r},\varepsilon'$ allowed under the return time parametrization of (10). The final r factors can, with impunity, be taken to right fill with $\varepsilon'=0$.

Equation (9) now says that each such amplitude is raised to power β, multiplied by z, and every allowed set of such products on a graph expressing "legal" transition is to be formed and summed. The result is then $1/u$, where u is that polynomial in z so constructed. Formulas (7) and (8) now determine $F(\beta)$.

The graph depicts a transition matrix T labeled by the allowed states of $\varepsilon_n,\ldots,\varepsilon_{n-r}$. The sum of successive products of n factors now produces $(zT)^n$. The sum over n is thus

$$\sum_m (1-zT)^{-1}_{m0} \propto 1/\det(1-zT)$$

where 0 stands for the state 0,..., 0, and so

$$u=\det(1-zT) \tag{12}$$

By (8), $z^{-1}(\beta)$ is an eigenvalue of T, and (7) is the canonical result.

Thus, once the transition graph is drawn, F is determined within combinations of a set of parameters that are the values $\sigma(\varepsilon_n,\ldots,\varepsilon_{n-r})$ in the rth approximation. If a theory for σ exists, F is determined. If not, we have a phenomenological theory of appropriate F's parametrized by combinations of the unknown σ's.

3. GOLDEN MEAN ROTATION

Let us compute F for golden mean rotation. This means that in Eq. (10),

$$T_m=F_m$$

where

$$F_{m+1}=F_m+F_{m-1} \tag{13}$$

It follows that a unique representation of t in (10) is had with $\varepsilon_i=0,1$, where $\varepsilon_{i+1}=1\rightarrow\varepsilon_i=0$.

The first approximation to σ consists of two ε's, and the values

$$\sigma(0,0)=\sigma_1,\qquad \sigma(0,1)=\sigma_2,\qquad \sigma(1,0)=\sigma_3$$

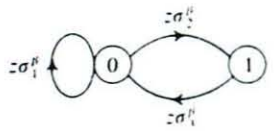

Figure 1

The transition graph is shown in Fig. 1. Our goal is to find the sum of paths in Fig. 1 starting and ending at any one node, since the denominator is always u of (12) independent of the node. The graph is manipulated by the elementary rules shown in Fig. 2.

It follows by inspection that

$$0=u_1=1-z\sigma_1^\beta-z^2(\sigma_2\sigma_3)^\beta \tag{14}$$

where u_1 denotes the first approximation. Observe that u_1 depends on just two parameters, σ_1 and $\sigma_2\sigma_3$. In fact, it depends on just σ_1. To see this, recall that

$$F(d_H)=0 \tag{15}$$

where d_H is the Hausdorff dimension. Since rotation covers the entire circle, $d_H=1$. According to (7), (15) implies that (14) is satisfied with $\beta=z=1$, so that

$$0=1-\sigma_1-\sigma_2\sigma_3 \tag{16}$$

and (14) becomes

$$0=1-z\sigma_1^\beta-z^2(1-\sigma_1^\beta) \tag{17}$$

Denoting the golden mean by ρ $[=(\sqrt{5}-1)/2]$, $N_n=F_{n+1}\sim\rho^{-n}$, and by (7),

$$F_1(\beta)=\beta\frac{\ln\sigma_1}{\ln\rho}+\frac{1}{\ln\rho}\ln\left(\left\{1+\left[1+4\left(\frac{1-\sigma_1}{\sigma_1 2}\right)^\beta\right]^{1/2}\right\}\Big/2\right) \tag{18}$$

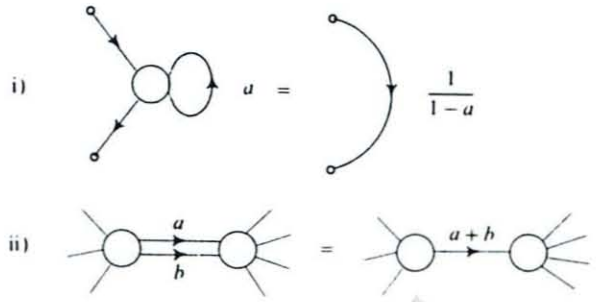

Figure 2

For subcritical rotation, Ref. 5 shows that our first approximation is exact with $\omega_1 = \rho$, so that

$$F_{\text{sub}}(\beta) = \beta - 1$$

and the f versus α curve degenerates to a point $f = \alpha = 1$. For criticality $\sigma_1 \simeq 0.47$ and F is nontrivial, with a qualitatively correct f versus α curve. Let us write down the second approximation before commenting.

We now have the legal transitions

$$000, 001, 010, 100, 101 \to \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5$$

and the transition graph (dual to Fig. 1) is depicted in Fig. 3. Manipulation produces

$$0 = u_2 = 1 - z\sigma_1^\beta - z^2(\sigma_3\sigma_3)^\beta - z^3[(\sigma_2\sigma_3\sigma_4)^\beta - (\sigma_1\sigma_3\sigma_5)^\beta] \tag{19}$$

Employing $0 = u_2(1, 1)$, and defining

$$s_1 \equiv \sigma_1, \qquad s_2 \equiv \sigma_3\sigma_5 \tag{20}$$

$$0 = u_2 = (1 - zs_1^\beta)(1 - z^2 s_2^\beta) - z^3(1 - s_1)^\beta(1 - s_2)^\beta \tag{21}$$

By Ref. 5, $s_2 > s_1$, and $s_1 s_2 > (1 - s_1)(1 - s_2)$, so that

$$z^{-1} \underset{\beta \to -\infty}{\sim} s_1^\beta, \qquad z^{-1} \underset{\beta \to +\infty}{\sim} (\sqrt{s_2})^\beta \tag{22}$$

According to the notation of Ref. 2, we thus have

$$\alpha_{\min} = \frac{\ln \rho}{\ln s_1}, \qquad \alpha_{\max} = 2\frac{\ln \rho}{\ln s_2} \tag{23}$$

One can now go ahead to successively higher order. Rather, let us quickly state results about arbitrary (infinite) order. It is easy to see from (12) that the coefficient of z^n in u is the sum of all independent n-cycles on the graph that are not decomposable into products of lower-order cycles. This contribution receives a minus sign. In addition, one adds the products of all lower-order disjoint cycles (each with a minus sign to form the product) for which the sum of lengths is n. From the form of legal states, it is easy to see that there is a unique cycle of length 1 for 0, 0,..., 0,..., and a

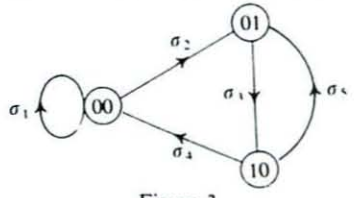

Figure 3

unique cycle of length 2 for 0, 1, 0, 1,... → 1, 0, 1, 0,... . Thus, s_1 and s_2 can be written down to any order. By RG calculations of Ref. 5,

$$s_1 \sim |\alpha|^{-3} \qquad \text{and} \qquad s_2 \sim |\alpha|^{-2} \tag{24}$$

with $\alpha^{-1} = -0.77...$ in criticality.

It is interesting in (24) that $\sqrt{s_2} \sim |\alpha|^{-1}$ as $n \to \infty$, because s_2 is always the product of two σ's, where, by Ref. 5, α^{-1} is *not* a dynamical scaling factor. That is, σ is well approximated as having three constant values so that (16) is satisfied, and $s_1 \sim \alpha^{-3}$, $s_2 \sim \alpha^{-6}$, with α^{-1} *not* a level-to-level successive refinement. It is precisely the degeneracy of F over internal scales that produces the intuitive maximum scale at α^{-1}. But this is a *real* phenomenon: with finite data of F_{n+1} points, there are n scaling actions, with the maximum scale set by the n-epsilon approximation. One can verify that at this level, s_2 has converged to α^{-2} to within α^{-n}. Since $\alpha^{-1} \sim 0.77$, $\alpha_{\max}$ of Ref. 2 for relatively large data sets will be measurably below the asymptotic prediction, and its f versus α curve that of u_n and not that of u_∞.

4. DISCUSSION AND CONCLUSIONS

Setting s_1 and s_2 in (21) to the asymptotic values (24) produces an f versus α curve that changes within 1% to the next level of calculation, and in excellent agreement with the numerical curve in Ref. 2. We can obviously compute it to any degree of accuracy. However, Fig. 1 produces a qualitatively correct result already. Let us say why this is so.

The lowest order period doubling calculation is shown in Fig. 4, so that

$$u = 1 - z(\sigma_1^\beta + \sigma_2^\beta) \tag{25}$$

With $\sigma_1 < \sigma_2$,

$$\alpha_{\min} = \frac{\ln 1/2}{\ln \sigma_1}, \qquad \alpha_{\max} = \frac{\ln 1/2}{\ln \sigma_2} \tag{26}$$

To all orders, this process always has two 1-cycles, that of 0000.00... and 11...1..., so that f versus α is fundamentally symmetric in its small- and large-scale behaviors. In contrast, the large scale in (14) is determined by the z^2 square term from its 2-cycle, and is hence asymmetric in its behavior.


Figure 4

All f versus α curves loosely resemble one another. They qualitatively differ most noticeably through the differences in small and large scales. It takes just a few numerical parameters to quantitatively fit experimental data once the parametric form of F is available. We now see that the crux of this form is set by the short-length cycles on the Markov graphs. This cycle distribution, however, is determined by the properties of close return times. Thus, f versus α is a data processing that principally comments upon return time properties of dynamical systems. Knowledge of return times determines the form of f versus α with parameters that can then be extracted. It would be most interesting to reverse this inference for more poorly understood systems.

In some perspective, the return time expansion of (10) determines at level n a quite large graph. The graph in good approximation reduces in size if well-behaved scalings can approximate (11). Now it is quite clear that low ε's cannot be very significant, since they represent a small number of iterates of the smooth dynamical process, which thus preserves the ratio of the small differences of (11). That there should be an exponential decrease, while true for period doubling and golden mean rotation, need not be generally true. However, there can still be a large class of interesting problems for which the short cycles on the graph predominantly mark the form of $F(\beta)$. With positive Liapunov exponents, there is an effective noise, which can move an orbit to a nearby point with different scaling properties. One might incorporate this by adding "stochastic" links on the graph that differ from the scaling links by having no z weights. It will thus be very interesting to see which—if any—of these ideas can be extracted from numerics on quite chaotic systems.

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Intrinsic Fluctuations in

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A reaction is called "explosiv
in a finite time. When the int
into account, the explosion
probability distribution, or at
reactions. If a reaction is uns
appear, one first has to comp
and then the average explosio
general Markov processes.

KEY WORDS: Explosions

1. INTRODUCTION

Consider chemical reactions with
x varies with time according to s

$$dx/$$

Until Section 5 we suppose

$$f(x) > 0$$

The solution of (1), with initial val

$$t =$$

If this integral converges as $x \to \infty$
finite time and we call the reactio

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