

## Problem Corner

### Solutions to Problems from February 2012

Provided by Shelomovskii Vladimir, Russia  
Interactive illustrations produced with GInMA [1]

**Question 1** Prove the inequality  $f(x, y, z) = 2x - 3y - 2z - 11 + 5\sqrt{x^2 + y^2 + 11 + 4z^2} \geq 0$ , and verify when the equality holds. Use CAS, if it is convenient.

**Solution.** It is known [2] that [root mean square](#) is greater or equal to [arithmetic mean](#)

$$\sqrt{\frac{a_1^2 + a_2^2 + \dots + a_n^2}{n}} \geq \frac{a_1 + a_2 + \dots + a_n}{n} \quad \text{and} \quad \sqrt{\frac{a_1^2 + a_2^2 + \dots + a_n^2}{n}} = \frac{a_1 + a_2 + \dots + a_n}{n} \quad \text{if and only if} \\ a_1 = a_2 = \dots = a_n. \quad (1)$$

If we set  $a_1 = 2z, a_2 = a_3 = a_4 = a_5 = -\frac{x}{2}, a_6 = \dots = a_{14} = \frac{y}{3}, a_{15} = \dots = a_{25} = 1$ .

$$\text{Then } \frac{a_1 + a_2 + \dots + a_n}{n} = \frac{2z + 4\left(-\frac{x}{2}\right) + 9\frac{y}{3} + 11 \cdot 1}{25} = \frac{-2x + 3y + 2z + 11}{25}.$$

$$\text{So } \sqrt{\frac{a_1^2 + a_2^2 + \dots + a_n^2}{n}} = \sqrt{\frac{(2z)^2 + 4 \cdot \left(-\frac{x}{2}\right)^2 + 9 \cdot \left(\frac{y}{3}\right)^2 + 11 \cdot 1^2}{25}} = \frac{\sqrt{4z^2 + x^2 + y^2 + 11}}{5}.$$

In view of (1) we obtain  $-\frac{x}{2} = \frac{y}{3} = 2z = 1$  or  $x = -2, y = 3, z = \frac{1}{2}$ .

#### CAS solution.

We make use of Maple to obtain the solution; however, we note that Maple cannot solve the inequality immediately in this case, we replace the inequality with the equality in our problem instead.

**restart; f:=5\*sqrt(x^2+y^2+4\*z^2+11)-(11+2\*z+3\*y-2\*x); s:=solve(f=0,x);**

$$s := -\frac{4z}{21} - \frac{2y}{7} - \frac{22}{21} - \frac{5\sqrt{-80z^2 + 12zy + 44z - 12y^2 + 66y - 110}}{21}, \\ -\frac{4z}{21} - \frac{2y}{7} - \frac{22}{21} + \frac{5\sqrt{-80z^2 + 12zy + 44z - 12y^2 + 66y - 110}}{21}$$

We understand that the radical in the Maple output above is greater or equal to zero. We again solve the equality instead of the inequality.

**g:=-80\*z^2+12\*z\*y+44\*z-12\*y^2+66\*y-110; so:=solve(g=0,y);**

$$so := \frac{z}{2} + \frac{11}{4} - \frac{1}{12}I\sqrt{231} + \frac{1}{6}I\sqrt{231}z, \frac{z}{2} + \frac{11}{4} + \frac{1}{12}I\sqrt{231} - \frac{1}{6}I\sqrt{231}z$$

Is it the solution in real numbers? Yes, but only if  $z = \frac{1}{2}$ .

**zz:=1/2;y:=subs(z=zz,so[1]);x:=subs(z=zz,s[1]);**

$$zz := \frac{1}{2} \quad y := 3 \quad x := -2$$

**Answer.**  $x = -2, y = 3, z = \frac{1}{2}$ .

**Question 2** Let  $f(x, y, a) = 1 - \sqrt{x^2 - 2axy + y^2 - 4y + 5}$ . Suppose the expression gets its maximum value only for single pair of  $(x, y)$ . Find the range of the parameter  $a \in [-1, 1]$ .

**Solution.** Let  $a^2 = 1$ . Then  $f(x, y, a) = 1 - \sqrt{(x - ay)^2 - 4y + 5} \leq 1$ . The function gets its maximum value  $f(x, y, a) = 1$  at several points. For example  $x = 2a \pm \sqrt{3}, y = 2$ .

For  $a^2 \neq 1$  we make transformation

$$x^2 - 2axy + y^2 - 4y + 5 = x^2 - 2axy + (ay)^2 + (1 - a^2)y^2 - 4y + 5 = (x - ay)^2 + (1 - a^2)y^2 - 2 \cdot 2y + \frac{2^2}{1 - a^2} - \frac{4}{1 - a^2} + 5 = (x - ay)^2 + \left(\sqrt{1 - a^2}y - \frac{2}{\sqrt{1 - a^2}}\right)^2 + \frac{1 - 5a^2}{1 - a^2}. \quad (2)$$

The maximum value of the function  $f(x, y, a) = 1 - \sqrt{x^2 - 2axy + y^2 - 4y + 5}$  occurs at the minimum of  $f_1(x, y, a) = \sqrt{x^2 - 2axy + y^2 - 4y + 5} = \sqrt{(x - ay)^2 + \left(\sqrt{1 - a^2}y - \frac{2}{\sqrt{1 - a^2}}\right)^2 + \frac{1 - 5a^2}{1 - a^2}}$ .

We consider the radical under the function  $f_1(x, y, a)$  in the following two cases:

**Case 1.** If  $a \in \left[-\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right]$  then  $\frac{1 - 5a^2}{1 - a^2} \geq 0$ . The smallest value of the radicand is achieved at a single

point, where both squares are equal to zero, that is: 
$$\begin{cases} \sqrt{1 - a^2}y - \frac{2}{\sqrt{1 - a^2}} = 0, \\ x - ay = 0, \end{cases} \Leftrightarrow \begin{cases} x = \frac{2a}{1 - a^2} \\ y = \frac{2}{1 - a^2} \end{cases}. \text{ Consequently,}$$

the function  $f(x, y, a) = 1 - \sqrt{x^2 - 2axy + y^2 - 4y + 5}$  obtain its maximum value at  $1 - \sqrt{\frac{1 - 5a^2}{1 - a^2}}$ .

**Case 2.** If  $a \in \left(-1; -\frac{1}{\sqrt{5}}\right) \cup \left(\frac{1}{\sqrt{5}}; 1\right)$  then  $\frac{1 - 5a^2}{1 - a^2} < 0$ . The smallest value of the radical under  $f_1(x, y, a)$  is achieved at every point where the radical is equal to zero, for example, at points

$$y = \frac{2}{1 - a^2}, \quad (x - ay)^2 + \frac{1 - 5a^2}{1 - a^2} = 0 \Leftrightarrow x = ay \pm \sqrt{\frac{5a^2 - 1}{1 - a^2}}. \text{ See Figures 1* - 2*}.$$

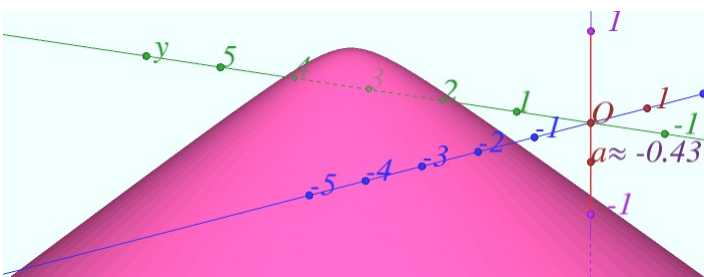


Figure 1\*: Solution for  $a \in \left[-\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right]$ .

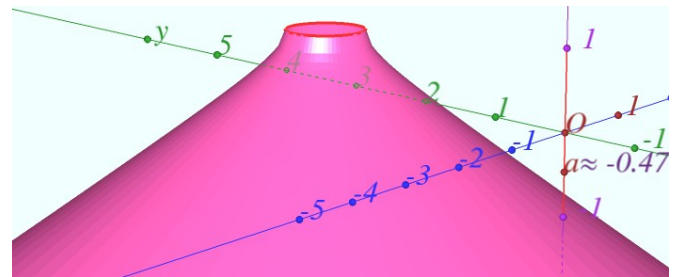


Figure 2\*: Solution for  $|a| > \frac{1}{\sqrt{5}}$ .

**Answer.**  $a \in \left[-\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right]$ .

**Question 3** Let  $N$  be the Nagel point of triangle  $ABC$ . Let  $A', B'$  and  $C'$  be the extouch points at which the  $A$ - excircle meets line  $BC$ , the  $B$ - excircle meets line  $AC$ , and  $C$ - excircle meets line  $AB$ , respectively. Let  $D$  be a point at  $AA'$  such that  $AD = NA'$ . Let  $E$  be a point at  $BB'$  such that  $BE = NB'$ . Let  $F$  be a point at  $CC'$  such that  $CF = NC'$ . Find the circumcenter of  $EFD$  triangle.

**Solution.** Let  $A'', B''$  and  $C''$  be tangent points of the inscribed circle of the triangle  $ABC$  with the sides  $BC$ ,  $AC$  and  $AB$ , respectively, see Figure 3\*. Let  $AC''=AB''=x$ ,  $BC''=BA''=y$ ,  $CA''=CB''=z$ . It is obvious that  $x+y+z=p$ , where  $p$  is a half-perimeter of  $ABC$  triangle. We also know that  $AB'=CB''=z$  and  $AC'=BC''=y$  (see [3]). Hence  $CB'=AB''=AC''=BC'=x$ .

Using the properties of intersecting segments, see Figure 4\*, we find that the segments  $BB'$  and  $CC'$  divide  $AA'$  at the same ratio  $\frac{A'N}{AA'}=\frac{x}{p}$ . So three segments  $AA'$ ,  $BB'$ , and  $CC'$  are collinear, and they cross at the Nagel point  $N$ . It is known that under the homothety with center at  $A$  and the coefficient  $\frac{x}{p}$ , the  $A$ - excircle passes into the inscribed circle. Hence  $\frac{AD}{AA'}=\frac{x}{p}$  and  $AD=A'N$ . The circle  $DEF$  is a circumcircle of  $DEF$  triangle and the inscribed circle of  $ABC$  triangle. The center of  $DEF$  circle is at the center of the inscribed circle of  $ABC$  triangle.

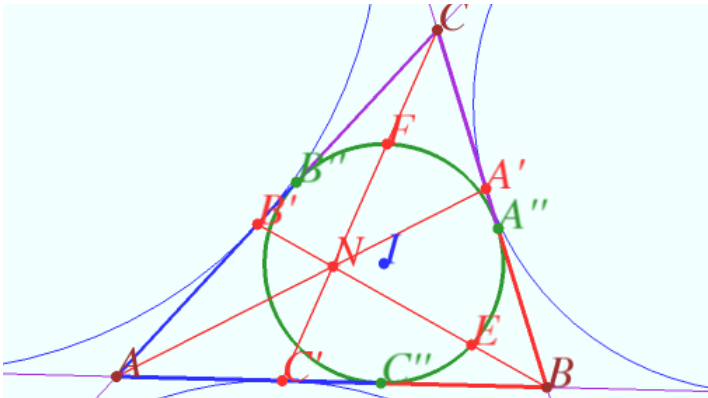


Figure 3\*: Nagel point characteristics

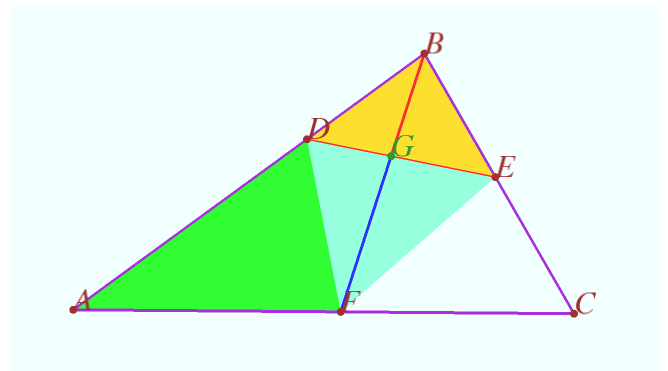


Figure 4\*: Intersecting segments

**Answer.** The circumcenter of  $EFD$  triangle is the incenter of  $ABC$  triangle.

\* **Note.** The Figures 1\* through 4\* are interactive. They will turn a static figure into an interactive one by following the instructions below.

The interactive figures are working if your computer has installed GInMA [1] program. To use the interactive figures in this note and several eJMT papers, it is enough to have GInMA demo version.

You can download GInMA demo from <http://www.deoma-cmd.ru/en/Products/Geometry/GInMA.aspx>. Here you can also download “GInMA user manual” with the instructions how to install GInMA, including demo, and the description of all the possibilities of interactive research using GInMA files.

After clicking the figures, a dialog box appears and asks for a permission to open the file; choose «allow». You must agree to give the program administrator rights if you are prompted to. Then the interactive draft opens appropriate to the figure from the paper on which you clicked. To return to the paper you are viewing, you must close the file with the interactive draft.

See video “How to turn static geometric figures into Interactive illustrations” [4].

## References

[1] [www.deoma-cmd.ru](http://www.deoma-cmd.ru).

[2] [http://en.wikipedia.org/wiki/Generalized\\_mean](http://en.wikipedia.org/wiki/Generalized_mean)

[3] [http://en.wikipedia.org/wiki/Incircle\\_and\\_excircles\\_of\\_a\\_triangle](http://en.wikipedia.org/wiki/Incircle_and_excircles_of_a_triangle)

[4] [http://youtu.be/g7KA-HzU\\_h0](http://youtu.be/g7KA-HzU_h0)