PROBLEM CORNER

Provided by Alvaro Gamboa
I.E.S Ciudad de los Poetas, Madrid, Spain
E-mail: alvarogamboa1245@gmail.com

Although Problem 2 has been proposed by prof. Ricardo Barroso (who has also proposed the corresponding solution), we present here the solution to Problems 1 and 2 that has been written by the proposer of Problem 1, namely, by the 16-years old student Alvaro Gamboa, to whom we would like to thank and to encourage getting so interested in doing mathematics!

Throughout this Solution, we will denote $r(A,B)$ the line passing through two different points *A* and *B*. In addition, $\overline{AB}$ will denote the segment joining *A* and *B*, whose length will be denoted by *AB*.

Problem 1. Let $X_1, X_2, \ldots, X_n$ be the vertices of a regular *n*-gon $\mathcal{P}$ and let *P* be any point interior to $\mathcal{P}$. We denote by P_{ij} the projection of *P* onto $r(X_i, X_j)$. By abuse, we denote $X_{n+1} := X_1$, and so $P_{n,1} := P_{n,n+1}$. Prove that the sum

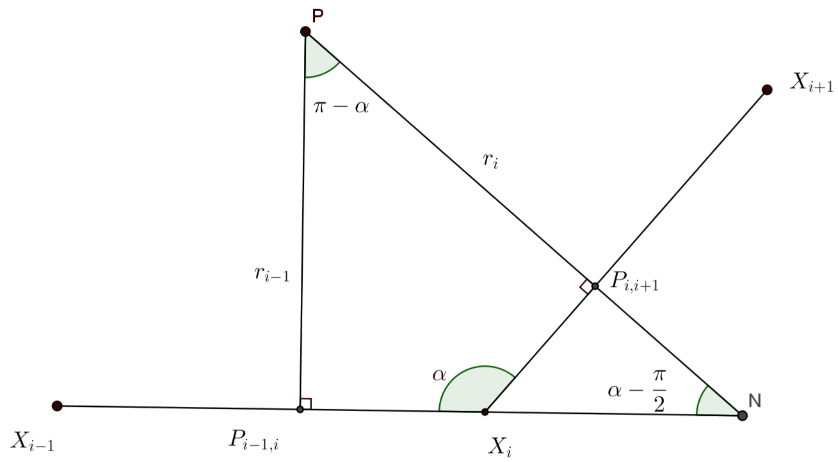
$$\sum_{i=1}^{n} X_i P_{i,i+1}$$

is constant, that is, it does not depend on the point *P*.

SOLUTION. We are going to prove that

$$\sum_{i=1}^{n} X_i P_{i,i+1} = s, \tag{1}$$

where *s* is the semi-perimeter of $\mathcal{P}$.



Let α be the measure of the angle formed by two whichever consecutive sides of $\mathcal{P}$. Also, let

$r_i := PP_{i,i+1}$ for every $i \in \{1,2,...,n\}$. For notational convenience we denote $r_0 := r_n$. Now, let us prove the following

Claim:

$$X_i P_{i,i+1} = \frac{r_{i-1} + r_i \cdot \cos\alpha}{\sin\alpha}, \text{ for every } i \in \{1,2,...,n\} \tag{2}$$

To prove it, let $N := r(P, P_{i,i+1}) \cap r(X_{i-1}, X_i)$. The measures of the angles of the quadrilateral $PP_{i,i+1}X_iP_{i-1,i}$ add up 2π, so $\measuredangle P_{i-1,i}PN = \pi - \alpha$, hence $\measuredangle PNP_{i-1,i} = \alpha - \frac{\pi}{2}$. Therefore,

$$-\cos\alpha = \cos(\pi - \alpha) = \cos\measuredangle PNP_{i-1,i} = \frac{r_{i-1}}{r_i + P_{i,i+1}N}$$

As a result,

$$P_{i,i+1}N = \frac{r_{i-1} + r_i \cdot \cos\alpha}{-\cos\alpha} \tag{3}$$

Furthermore,

$$\frac{-\cos\alpha}{\sin\alpha} = \tan\left(\alpha - \frac{\pi}{2}\right) = \tan(\measuredangle P_{i,i+1}NX_i) = \frac{X_iP_{i,i+1}}{P_{i,i+1}N} \tag{4}$$

From (3) and (4) one gets $r_{i-1} + r_i \cdot \cos\alpha = X_iP_{i,i+1} \cdot \sin\alpha$, which proves Claim (2).

In light of the Claim, statement (1) is equivalent to

$$\sum_{i=1}^{n} \frac{r_{i-1} + r_i \cdot \cos\alpha}{\sin\alpha} = s \tag{5}$$

Before proving this equality, let us make some additional observations.

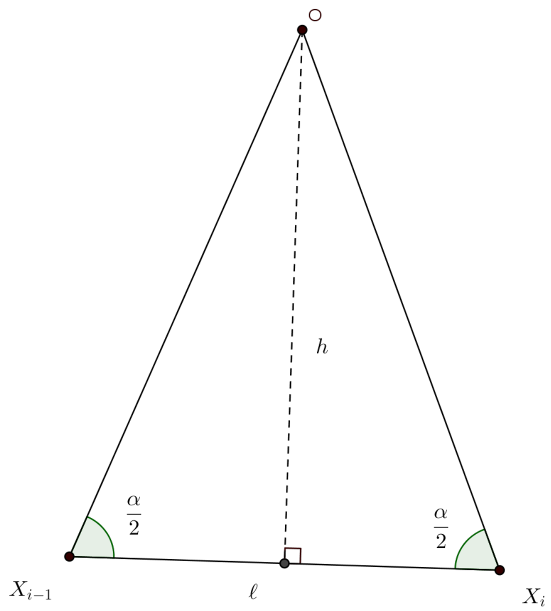
Let 1 be the length of each side of P. Now, we will express the area $\mathrm{A}(\mathrm{P})$ of P in two different ways. On one hand,

$$\mathrm{A}(\mathrm{P}) = \sum_{i=1}^{n} \mathrm{A}_{\Delta PX_iX_{i+1}} = \sum_{i=1}^{n} \frac{1 \cdot r_i}{2} = \frac{1}{2} \cdot \sum_{i=1}^{n} r_i \tag{6}$$

On the other hand, it is well known that

$$\mathrm{A}(\mathrm{P}) = s \cdot h, \tag{7}$$

where h is the distance from the center of P to any side of P.



Notice that

$$\tan\left(\frac{\alpha}{2}\right)=\frac{h}{\frac{1}{2}},\ \text{ and so }\ h=\frac{1}{2}\cdot\tan\left(\frac{\alpha}{2}\right)$$

Substituting this value of h into (7) yields

$$\mathrm{A}\left(\mathrm{P}\right)=s\cdot\frac{1}{2}\cdot\tan\left(\frac{\alpha}{2}\right) \tag{8}$$

Substracting (6) and (8) gives

$$\sum_{i=1}^{n} r_i = s\cdot\tan\left(\frac{\alpha}{2}\right) \tag{9}$$

We are ready to prove (5) and so the statement.

$$\sum_{i=1}^{n}\frac{r_{i-1}+r_i\cdot\cos\alpha}{\sin\alpha}=\frac{(r_0+r_1\cdot\cos\alpha)+(r_1+r_2\cdot\cos\alpha)+\cdots+(r_{n-1}+r_n\cdot\cos\alpha)}{\sin\alpha}$$

$$=\frac{(r_1\cdot\cos\alpha+r_1)+(r_2\cdot\cos\alpha+r_2)+\cdots+(r_n\cdot\cos\alpha+r_n)}{\sin\alpha}$$

$$=\frac{(r_1+r_2+\cdots r_n)\cdot(\cos\alpha+1)}{\sin\alpha}=\left(\sum_{i=1}^{n} r_i\right)\cdot\left(\frac{\cos\alpha+1}{\sin\alpha}\right)$$

$$\overset{(9)}{=} s\cdot\tan\left(\frac{\alpha}{2}\right)\cdot\left(\frac{\cos\alpha+1}{\sin\alpha}\right) = s\cdot\tan\left(\frac{\alpha}{2}\right)\cdot\left(\frac{\cos^2\left(\frac{\alpha}{2}\right)-\sin^2\left(\frac{\alpha}{2}\right)+1}{2\sin\left(\frac{\alpha}{2}\right)\cdot\cos\left(\frac{\alpha}{2}\right)}\right)$$

$$= s\cdot\frac{\sin\left(\frac{\alpha}{2}\right)}{\cos\left(\frac{\alpha}{2}\right)}\cdot\left(\frac{2\cos^2\left(\frac{\alpha}{2}\right)}{2\sin\left(\frac{\alpha}{2}\right)\cdot\cos\left(\frac{\alpha}{2}\right)}\right) = s,$$

as claimed.

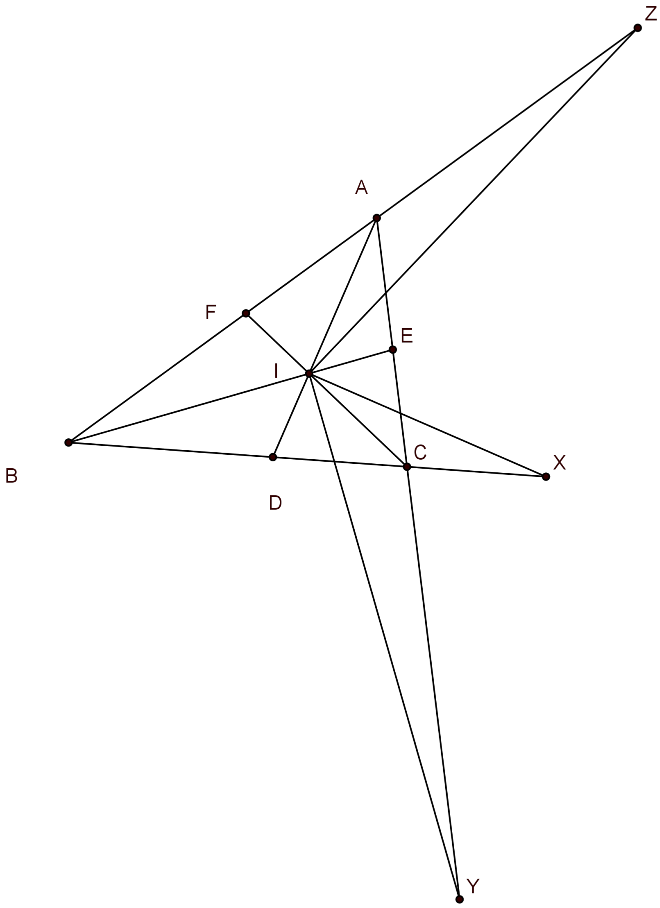
Problem 2. Let I be the incenter of a triangle ΔABC, that is, the point of intersection of the bisectors of the angles of the triangle. Let l_1, l_2 and l_3 be, respectively, the lines which are perpendicular through I to the lines $r(A,I)$, $r(B,I)$ and $r(C,I)$. Prove that the points

$$X := r(B,C)\cap l_1,\ Y := r(A,C)\cap l_2 \text{ and } Z := r(A,B)\cap l_3.$$

are collinear.

SOLUTION. By Menelao's theorem it is enough to prove the equality

$$\frac{XB}{XC}\cdot\frac{YC}{YA}\cdot\frac{ZA}{ZB} = 1 \tag{1}$$



Consider the auxiliary points

$$D := r(A,I) \cap r(B,C), \quad E := r(B,I) \cap r(A,C) \quad \text{and} \quad F := r(C,I) \cap r(A,B).$$

Let us denote

$$\measuredangle BAC := 2\alpha, \quad \measuredangle CBA := 2\beta \quad \text{and} \quad \measuredangle ACB = 2\gamma$$

Since I is the incenter of ΔABC the following equalities hold:

$$\measuredangle BAI = \measuredangle IAC = \alpha, \quad \measuredangle CBI = \measuredangle IBA = \beta \quad \text{and} \quad \measuredangle ACI = \measuredangle ICB = \gamma$$

Consequently,

$$\measuredangle IDB = \pi - \alpha - 2\beta, \quad \measuredangle IEC = \pi - \beta - 2\gamma, \quad \measuredangle IFA = \pi - \gamma - 2\alpha,$$

$$\measuredangle CDI = \pi - \alpha - 2\gamma, \quad \measuredangle AEI = \pi - \beta - 2\alpha, \quad \measuredangle BFI = \pi - \gamma - 2\beta$$

Therefore,

$$\measuredangle BID = \alpha + \beta = \measuredangle EIA, \quad \measuredangle CIE = \beta + \gamma = \measuredangle FIB \quad \text{and} \quad \measuredangle AIF = \alpha + \gamma = \measuredangle DIC$$

As a result,

$$\measuredangle CIX = \frac{\pi}{2} - (\alpha+\gamma), \quad \measuredangle YIC = \frac{\pi}{2} - (\beta+\gamma), \quad \measuredangle ZIA = \frac{\pi}{2} - (\alpha+\gamma),$$

$$\measuredangle IXC = \alpha + 2\gamma - \frac{\pi}{2}, \quad \measuredangle CYI = \beta + 2\gamma - \frac{\pi}{2} \quad \text{and} \quad \measuredangle AZI = 2\alpha + \gamma - \frac{\pi}{2}$$

Applying the Law of sines to triangles ΔXIC and ΔXBI, one respectively gets

$$\frac{XC}{\sin\left(\frac{\pi}{2} - (\alpha+\gamma)\right)} = \frac{IX}{\sin\gamma} \quad \text{and} \quad \frac{XB}{\sin\left(\frac{\pi}{2} + (\alpha+\beta)\right)} = \frac{IX}{\sin\beta}$$

Applying the Law of sines to triangles ΔYIC and ΔYAI, one respectively gets

$$\frac{YC}{\sin\left(\frac{\pi}{2} - (\beta+\gamma)\right)} = \frac{IY}{\sin\gamma} \quad \text{and} \quad \frac{YA}{\sin\left(\frac{\pi}{2} + (\alpha+\beta)\right)} = \frac{IY}{\sin\alpha}$$

and, applying the Law of sines to triangles ΔZIA and ΔZBI, one respectively gets

$$\frac{ZA}{\sin\left(\frac{\pi}{2} - (\alpha+\gamma)\right)} = \frac{IZ}{\sin\alpha} \quad \text{and} \quad \frac{ZB}{\sin\left(\frac{\pi}{2} + (\beta+\gamma)\right)} = \frac{IY}{\sin\beta}$$

Finally we obtain

$$\frac{XB}{XC}\cdot\frac{YC}{YA}\cdot\frac{ZA}{ZB} = \frac{\sin\left(\frac{\pi}{2} + (\alpha+\beta)\right)}{\sin\left(\frac{\pi}{2} - (\alpha+\gamma)\right)}\cdot\left(\frac{\sin\gamma}{\sin\beta}\right)\cdot\frac{\sin\left(\frac{\pi}{2} - (\beta+\gamma)\right)}{\sin\left(\frac{\pi}{2} + (\alpha+\beta)\right)}\cdot\left(\frac{\sin\alpha}{\sin\gamma}\right)$$

$$\cdot\frac{\sin\left(\frac{\pi}{2} - (\alpha+\gamma)\right)}{\sin\left(\frac{\pi}{2} + (\beta+\gamma)\right)}\cdot\left(\frac{\sin\beta}{\sin\alpha}\right) = \frac{\sin\left(\frac{\pi}{2} - (\beta+\gamma)\right)}{\sin\left(\frac{\pi}{2} + (\beta+\gamma)\right)} = 1,$$

as claimed.