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A CONTINUUM HAVING ITS HYPERSPACES NOT LOCALLY CONTRACTIBLE AT THE TOP

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ABSTRACT. For a continuum X, let $C(X)$ (resp. 2^X) be the spaces of all nonempty subcontinua (resp. closed subsets) of X. In this paper we answer a question of Dilks by showing an example of a continuum X such that if $\mathscr{H} = C(X)$ or 2^X, then $\mathscr{H}$ does not have nonempty open subsets which are contractible in $\mathscr{H}$. In particular, $\mathscr{H}$ is not locally contractible at any of its points.

INTRODUCTION

A continuum is a compact, connected, nondegenerate metric space. The *hyperspaces* of a continuum X are the spaces 2^X consisting of all nonempty closed subsets of X, and $C(X)$, consisting of the connected elements in 2^X, each with the Hausdorff metric. If U is a subset of a topological space Y, we say that U is contractible in Y if there exists a continuous function $F: U \times I \to Y$ and there exists a point $y \in Y$ such that $F(u, 0) = u$ and $F(u, 1) = y$ for every $u \in U$.

In [4] the contractibility of hyperspaces is discussed in detail. More recent results may be found in [1], [2], [5–9]. In [3, Problem 111], appears the following question by Dilks: Is $C(X)$ or 2^X locally contractible at the point X? In this paper we give an example of a continuum X such that if $\mathscr{H} = C(X)$ or 2^X, then $\mathscr{H}$ does not have nonempty open subsets which are contractible in $\mathscr{H}$. In particular, $\mathscr{H}$ is not locally contractible at any of its points.

1. AN AUXILIARY CONSTRUCTION

Let $Q = [-1, 1] \times [-1, 1] \times \dots$ with the metric $d(x, y) = \sum |x_n - y_n|/(2^n)$, where $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$. Let $J = [-1, 1]$, and let $\mathbb{N} = \{1, 2, \dots\}$. For $n \in \mathbb{N}$, define the projection $\Pi_n: Q \to J^n$ by $\Pi_n(x) = (x_1, \dots, x_n)$. Given $x, y \in Q$, define the segment joining x and y by $\overline{xy} = \{tx + (1-t)y \in Q: t \in [0, 1]\}$.

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Given $n \in \mathbb{N}$, $A \subset J^n \times \{0\} \times \dots$, and $p, q \in A$ define, for $m \in \mathbb{N}$, $A_m = \Pi_n(A) \times \{1/m\} \times \{0\} \times \dots$, $p_m = (\Pi_n(p), 1/m, 0, \dots) \in A_m$, $q_m = (\Pi_n(q), 1/m, 0, \dots)$, and

$$L_m = \begin{cases} \overline{p_m p_{m+1}} & \text{if } m \text{ is even,} \\ \overline{q_m q_{m+1}} & \text{if } m \text{ is odd.} \end{cases}$$

Then we define $P(n, A, p, q) = A \cup (\bigcup\{A_m \cup L_m : m \in \mathbb{N}\})$ and $N(n, A, p, q) = \{(x_1, \dots, x_n, -x_{n+1}, x_{n+2}, \dots) \in Q : x = (x_1, x_2, \dots) \in P(n, A, p, q)\}$.

Since $A = \Pi_n(A) \times \{0\} \times \dots$, we have that $P(n, A, p, q) \subset \Pi_n(A) \times [0, 1] \times \{0\} \times \dots$. Then $\Pi_n(P(n, A, p, q)) = \Pi_n(A) = \Pi_n(N(n, A, p, q))$. Notice that $A = \mathrm{Cl}_Q(\bigcup\{A_m \cup L_m : m \in \mathbb{N}\}) - (\bigcup\{A_m \cup L_m : m \in \mathbb{N}\})$. So if A is a continuum, then $P(n, A, p, q)$ and $N(n, A, p, q)$ are continua. Notice also that $P(n, A, p, q) \cap (J^n \times \{0\} \times \dots) = A = N(n, A, p, q) \cap (J^n \times \{0\} \times \dots)$.

For $n \in \mathbb{N}$ and $u \in J$, define $Q(n, u) = J^n \times \{u\} \times J \times \dots$. If $m \in \mathbb{N}$ and $u \in (1/(m+1), 1/m)$, then $Q(n, u) \cap P(n, A, p, Q) = Q(n, u) \cap L_m$ is a point which separates $P(n, A, p, q)$.

2. The example X

Let $a = (-1, 0, \dots)$ and $b = (1, 0, \dots)$. For $n \in \mathbb{N}$, put $a_n = (-1 + 1/2^n, 0, \dots)$ and $b_n = (1 - 1/2^n, 0, \dots)$. Define

$$\begin{aligned} A^* &= \overline{a_1 b_1}; \\ B_1 &= P(1, A^*, a_1, b_1); \\ C_1 &= P(2, B_1 \cup \overline{a_2 b_1}, a_2, b_1), \\ D_1 &= N(2, B_1 \cup \overline{a_1 b_2}, a_1 . b_2). \end{aligned}$$

In general, for $n \geq 2$, define $B_n = P(2n-1, C_{n-1} \cup D_{n-1}, a_n, b_n)$; $C_n = P(2n, B_n \cup \overline{a_{n+1} b_n}, a_{n+1}, b_n)$ and $D_n = N(2n, B_n \cup \overline{a_n b_{n+1}}, a_n, b_{n+1})$.

Then define $X = \mathrm{Cl}_Q(\bigcup\{B_n : n \in \mathbb{N}\})$. Since A^* is a continuum, then B_1, C_1, and D_1 are continua. It follows that each B_n is a continuum. Furthermore $B_1 \subset C_1 \cap D_1 \subset B_2 \subset C_2 \cap D_2 \subset \dots$. So X is a continuum. We will prove some properties of X:

(A) $\qquad \Pi_{2n+1}(X) = \Pi_{2n+1}(C_n \cup D_n \cup \overline{ab})$ for all $n \in \mathbb{N}$.

Since $\overline{a_n b_n} \subset B_n$ and $\overline{a_n b_n} \to \overline{ab}$, we have that $\overline{ab} \subset X$. Then $C_n \cup D_n \cup \overline{ab} \subset X$. If $m > n+1$, then $\Pi_{2m-1}(B_m) = \Pi_{2m-1}(C_{m-1} \cup D_{m-1})$. This implies that

$$\begin{aligned} \Pi_{2m-2}(B_m) &= \Pi_{2m-2}(C_{m-1}) \cup \Pi_{2m-2}(D_{m-1}) \\ &= \Pi_{2m-2}(B_{m-1} \cup \overline{a_m b_{m-1}}) \cup \Pi_{2m-2}(B_{m-1} \cup \overline{a_{m-1} b_m}) \\ &\subset \Pi_{2m-2}(B_{m-1} \cup \overline{ab}). \end{aligned}$$

So $\Pi_{2m-3}(B_m) \subset \Pi_{2m-3}(B_{m-1} \cup \overline{ab})$. Then $\Pi_{2n+1}(B_m) \subset \Pi_{2n+1}(B_{m-1} \cup \overline{ab})$. It follows that

$$\Pi_{2n+1}(B_m) \subset \Pi_{2n+1}(B_{n+1} \cup \overline{ab}).$$

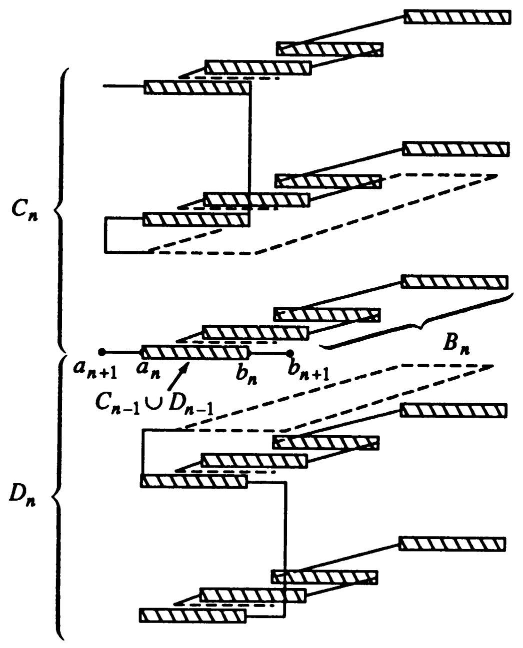

FIGURE 1

$$C_{n-1} \cup D_{n-1} J^{2n-1} \times \{0\} \times \dots$$
$$B_n \subset J^{2n-1} \times [0, 1] \times \{0\} \times \dots$$
$$C_n \subset J^{2n-1} \times [0, 1] \times [0, 1] \times \{0\} \times \dots$$
$$D_n \subset J^{2n-1} \times [0, 1] \times [-1, 0] \times \{0\} \times \dots$$

But

$$\Pi_{2n+1}(B_{n+1}) = \Pi_{2n+1}(C_n \cup D_n);$$

thus

$$\Pi_{2n+1}\left(\bigcup\{B_m : m > n\}\right) \subset \Pi_{2n+1}(C_n \cup D_n \cup \overline{ab}).$$

Since $B_1 \subset \cdots \subset B_n \subset C_n \cup D_n$ and $\Pi_{2n+1}(C_n \cup D_n \cup \overline{ab})$ is compact, we conclude that $\Pi_{2n+1}(X) = \Pi_{2n+1}(C_n \cup D_n \cup \overline{ab})$.

(B) $\Pi_{2n}(X) = \Pi_{2n}(B_n \cup \overline{ab})$ for all $n \in \mathbb{N}$, and the maps $r_1 : X \to C_n \cup D_n \cup \overline{ab}$ and $r_2 : X \to B_n \cup \overline{ab}$ given by $r_1(x) = (\Pi_{2n+1}(x), 0, \dots)$ and $r_2(x) = (\Pi_{2n}(x), 0, \dots)$ are retractions.

Notice that $\Pi_{2n}(X) \subset \Pi_{2n}(C_n) \cup \Pi_{2n}(D_n) \cup \Pi_{2n}(\overline{ab}) \subset \Pi_{2n}(B_n \cup \overline{a_{n+1}b_n}) \cup \Pi_{2n}(B_n \cup \overline{a_n b_{n+1}}) \cup \Pi_{2n}(\overline{ab}) \subset \Pi_{2n}(B_n \cup \overline{ab})$. Since $C_n \cup D_n \cup \overline{ab} \subset J^{2n+1} \times \{0\} \times \dots$, we have that $C_n \cup D_n \cup \overline{ab} = \Pi_{2n+1}(X) \times \{0\} \times \dots$. Then r_1 is a retraction. Similarly, r_2 is a retraction.

(C) $B_n \cap \overline{ab} = \overline{a_n b_n}$ for every $n \in \mathbb{N}$.

$B_1 \cap \overline{ab} = P(1, A^*, a, b) \cap (J \times \{0\} \times \dots) = A^* = \overline{a_1 b_1}$. Suppose that $B_n \cap \overline{ab} = \overline{a_n b_n}$. Then $B_{n+1} \cap \overline{ab} = P(2n+1, C_n \cup D_n, a_{n-1}, b_{n-1}) \cap (J^{2n-1} \times \{0\} \times \dots) \cap (J \times \{0\} \times \dots) = (C_n \cup D_n) \cap (J^{2n} \times \{0\} \times \dots) \cap (J \times \{0\} \times \dots) = ((B_n \cup \overline{a_{n+1} b_n}) \cup (B_n \cup \overline{a_n b_{n+1}})) \cap \overline{ab} = \overline{a_{n+1} b_{n+1}}$.

(D) Assume that $n \in \mathbb{N}$, $Y = C_n \cup D_n \cup \overline{ab}$ and $\mathscr{H}_0 = C(Y)$ or 2^Y. Then there do not exist $B \in \mathscr{H}_0$, W open in $\mathscr{H}_0$ and $F: W \times I \to \mathscr{H}_0$ continuous such that $B \subset B_n$, $B \in W$, for each $A \in W$, $F(A, 0) = A$, $F(A, 1) = Y$ and, for every $A \in W$ and $s \le t$, $F(A, s) \subset F(A, t)$.

Suppose that there exist such B, W, and F. Since $F(B, 0) \subset B_n$ and $F(B, 1) = Y \not\subset B_n$, there exists $t_0 = \max\{t \in [0, 1]: F(B, t) \subset B_n\}$ and $0 \le t_0 < 1$. By (C), $a_{n+1}, b_{n+1} \notin B_n$ and, since $B_n \subset J^{2n} \times \{0\} \times \dots$, we have that $B_n \cap ((\{\Pi_{2n}(a_{n+1})\} \times J \times \dots) \cup (\{\Pi_{2n}(b_{n+1})\} \times J \times \dots)) = \varnothing$. So there exists $\delta > 0$ such that $t_0 + \delta/2 < 1$; $\mathscr{B} = \{A \in \mathscr{H}_0 : H(A, B) < \delta\} \subset W$ (H is the Hausdorff metric for 2^Q) and, if $A \in \mathscr{B}$ and $t \in (t_0 - \delta, t_0 + \delta) \cap [0, 1]$, then $F(A, t) \cap ((\{\Pi_{2n}(a_{n+1})\} \times J \times \dots) \cup (\{\Pi_{2n}(b_{n+1}\} \times J \times \dots)) = \varnothing$.

We will prove that $F(B, t_0 + \delta/2) \subset C_n$. Recall that

$$\begin{aligned} C_n &= P(2n, B_n \cup \overline{a_{n+1} b_n}, a_{n+1}, b_n) \\ &= B_n \cup \overline{a_{n+1} b_n} \cup \left(\bigcup \{A_m \cup L_m : m \in \mathbb{N}\}\right) \end{aligned}$$

where $A_m = (\Pi_{2n}(B_n \cup \overline{a_{n+1} b_n})) \times \{1/m\} \times \{0\} \times \dots$;

$$L_m = \begin{cases} \overline{p_m p_{m+1}} & \text{if } m \text{ is even} \\ \overline{q_m q_{m+1}} & \text{if } m \text{ is odd,} \end{cases}$$
$$p_m = (\Pi_{2n}(a_{n+1}), 1/m, 0, \dots),$$
$$q_m = (\Pi_{2n}(b_n), 1/m, 0, \dots).$$

Since $B \subset B_n \subset J^{2n} \times \{0\} \times \dots$, we have that $B = \Pi_{2n}(B) \times \{0\} \times \dots$. For each $m \in \mathbb{N}$, put $E_m = \Pi_{2n}(B) \times \{1/m\} \times \{0\} \times \dots \subset A_m$. Then $E_m \in \mathscr{H}_0$ and $E_m \to B$, so there exists $M \in \mathbb{N}$ such that $H(B, E_m) < \delta$ for all $m \ge M$.

Let m be an even number with $m \ge M$. We will show that

$$T = F(E_m, t_0 + \delta/2) \subset C_n.$$

Choose $u \in (1/(m+1), 1/m)$. Then $Q(2n, u) \cap C_n = Q(2n, u) \cap \overline{p_m p_{m+1}}$. Since $\{F(E_m, s) \in 2^Y : 0 \le s \le t_0 + \delta/2\}$ is an order arc in 2^Y from $F(E_m, 0)$ to T, then [4, Theorem 1.8] each component of T intersects $E_m \subset \Pi_{2n}(B_n) \times \{1/m\} \times \{0\} \times \dots$. Thus $S = T \cup (\Pi_{2n}(B_n) \times \{1/m\} \times \{0\} \times \dots)$ is connected. Since $\overline{p_m p_{m+1}} \subset \{\Pi_{2n}(a_{n+1})\} \times J \times \dots$, $B_n \subset J^{2n} \times \{0\} \times \dots$, and $a_{n+1} \notin B_n$, we have that $S \cap \overline{p_m p_{m+1}} = \varnothing$. On the other hand, $S \subset Y = C_n \cup D_n \cup \overline{ab}$ and $D_n \cup \overline{ab} \subset J^{2n} \times [-1, 0] \times \{0\} \times \dots$, so $S \cap Q(2n, u) = S \cap Q(2n, u) \cap C_n = S \cap Q(2n, u) \cap \overline{p_m p_{m+1}} = \varnothing$. Furthermore S is connected and $E_m \subset S \cap J^{2n} \times (u, 1] \times J \times \dots$.

Then $S \subset (J^{2n} \times (u, 1] \times J \times \dots) \cap Y \subset C_n$. Thus $F(E_m, t_0 + \delta/2) \subset C_n$ for all $m \geq M$ with m even. Hence $F(B, t_0 + \delta/2) \subset C_n$.

In a similar way it may be proved that

$$F(B, t_0 + \delta/2) \subset D_n.$$

Then $F(B, t_0+\delta/2) \subset D_n$. Then $F(B, t_0+\delta/2) \subset C_n \cap D_n = C_n \cap (J^{2n} \times \{0\} \times \dots) \cap D_n = (B_n \cup \overline{a_{n+1} b_n}) \cap (B_n \cup \overline{a_n b_{n+1}}) = B_n$. Therefore $F(B, t_0+\delta/2) \subset B_n$. This contradicts the choice of t_0 and proves (D).

(E)
$$C(X) = \mathrm{Cl}_{c(x)} \left(\bigcup \{C(B_n) : n \in \mathbb{N}\} \right),$$

and

$$2^x = \mathrm{Cl}_{2^x} \left(\bigcup \{2^{Bn} : n \in \mathbb{N}\} \right).$$

For $n \in \mathbb{N}$, take the natural retraction $f\colon \overline{ab} \to \overline{a_n b_n}$ and define $r\colon X \to B_n$ by

$$r(x) = \begin{cases} f(r_2(x)) & \text{if } r_2(x) \in \overline{ab}, \\ r_2(x) & \text{if } r_2(x) \in B_n, \end{cases}$$

with r_2 as in (B). Then r is a retraction such that $D(x, r(x)) \leq 1/(2^{n-1})$ for every $x \in X$. Hence, for each $B \in 2^X$, $H(B, r(B)) \leq 1/(2^{n-1})$.

(F)

If $\mathscr{H} = C(X)$ or 2^X, then $\mathscr{H}$ does not have nonempty open subsets which are contractible in $\mathscr{H}$.

Suppose that there exist an open nonempty subset U of $\mathscr{H}$, a continuous function $G\colon U \times I \to \mathscr{H}$, and $A_0 \in \mathscr{H}$ such that $G(A, 0) = A$ and $G(A, 1) = A_0$ for every $A \in U$. Then the function $K\colon U \times I \to \mathscr{H}$ given by $K(A, t) = \bigcup \{G(A, s) : 0 \leq s \leq t\}$ is continuous [4, Lemma 16.3]. Let $\alpha\colon [1/2, 1] \to \mathscr{H}$ be a continuous function such that $\alpha(1/2) = A_0$, $\alpha(1) = X$ and $\alpha(s) \subset \alpha(t)$ if $s \leq t$. Choose $n \in \mathbb{N}$ such that $U \cap 2^{Bn} \neq \varnothing$. Let $Y = C_n \cup D_n \cup \overline{ab}$, $\mathscr{H}_0 = 2^Y \cap \mathscr{H}$, and $W = U \cap \mathscr{H}_0$. Fix $B \in W \cap 2^{Bn}$. Define $F\colon W \times I \to \mathscr{H}_0$ by:

$$F(A, t) = \begin{cases} r_1(K(A, 2t)) & \text{if } 0 \leq t \leq 1/2, \\ r_1(K(A, 1) \cup \alpha(t)) & \text{if } 1/2 \leq t \leq 1, \end{cases}$$

where r_1 is the retraction defined in (B) and $r_1(Z)$ means the image of Z under r. Properties of B, W, and F contradict property (D), and the contradiction completes the proof of (F).

Added in proof. The question answered in this paper has also been answered by Hisao Kato, *On local contractibility at X in hyperspaces $C(X)$ and 2^X*, Houston J. Math. **15** (1989), 363–370.

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