1.4 Inverses of Functions

Essential Question: What is an inverse function, and how do you know it's an inverse function?

Resource Locker

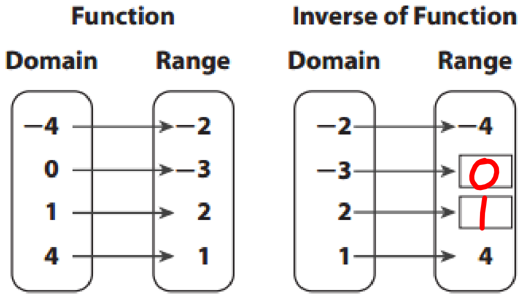
Explore Understanding Inverses of Functions

Recall that a *relation* is any pairing of the elements of one set (the domain) with the elements of a second set (the range). The elements of the domain are called inputs, while the elements of the range are called outputs. A function is a special type of relation that pairs every input with exactly one output. In a *one-to-one function*, no output is ever used more than once in the function's pairings. In a *many-to-one function*, at least one output is used more than once.

An **inverse relation** reverses the pairings of a relation. If a relation pairs an input x with an output y, then the inverse relation pairs an input y with an output x. The inverse of a function may or may not be another function. If the inverse of a function $f(x)$ is also a function, it is called the **inverse function** and is written $f^{-1}(x)$. If the inverse of a function is not a function, then it is simply an inverse relation.

(A) The mapping diagrams show a function and its inverse. Complete the diagram for the inverse of the function.

Is the function one-to-one or many-to-one? Explain.



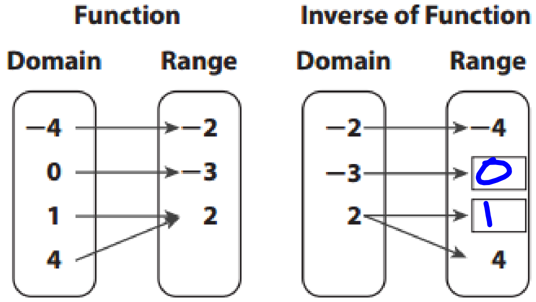
The function is one-to-one, because no output is ever used more than once in the function's pairings.

Is the inverse of the function also a function? Explain.

yes, each input has exactly one output

(B) The mapping diagrams show a function and its inverse. Complete the diagram for the inverse of the function.

Is the function one-to-one or many-to-one? Explain.

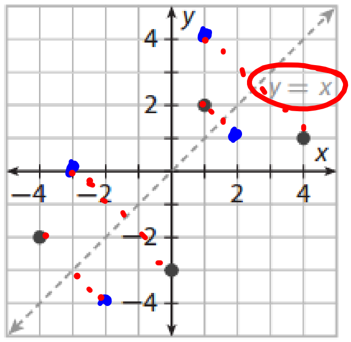


The function is many-to-one because inputs 1 and 4 have the same output 2

Is the inverse of the function also a function? Explain.

The inverse relation is not a function because the input 2 has two outputs, 1 and 4

(C) The graph of the original function in Step A is shown. Note that the graph also shows the dashed line $y = x$. Write the inverse of the function as a set of ordered pairs and graph them.



Function: $\{(-4, -2), (0, -3), (1, 2), (4, 1)\}$

Inverse of function:

$\{(-2, -4), (-3, 0), (2, 1), (1, 4)\}$

What do you observe about the graphs of the function and its inverse in relationship to the line $y = x$? Why does this make sense?

The graphs reflect across the line y=x, which makes sense because the ordered pairs in a function and its inverse have their x and y coordinates reversed

(D) The **composition of two functions** $f(x)$ and $g(x)$, written $f(g(x))$ and read as "f of g of x," is a new function that uses the output of $g(x)$ as the input of $f(x)$. For example, consider the functions f and g with the following rules.

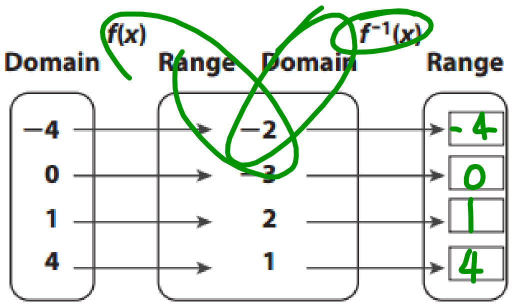
f: Add 1 to an input. g: Double an input. $f(x) = x + 1$ $g(x) = 2x$

Notice that $g(1) = 2(1) = 2$. So, $f(g(1)) = f(2) = 2 + 1 = 3$.

You can also find $g(f(x))$. Notice that $f(1) = 1 + 1 = 2$. So, $g(f(1)) = g(2) = 2(2) = 4$.

For these two functions, you can see that $f(g(1)) \neq g(f(1))$.

You can compose a function and its inverse. For instance, the mapping diagram shown illustrates $f^{-1}(f(x))$ where $f(x)$ is the original function from Step A and $f^{-1}(x)$ is its inverse. Notice that the range of $f(x)$ serves as the domain of $f^{-1}(x)$. Complete the diagram. What do you notice about the outputs of $f^{-1}(f(x))$? Explain why this makes sense.



The outputs of $f^{-1}(f(x))$ exactly match the inputs of $f^{-1}(f(x))$. This means $f^{-1}(f(x)) = X$. The original function takes an input and assigns an output. The inverse picks up that output and uses it as an input. The inverse turns around and has an output equal to the original input.

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Reflect

1. What is the relationship between the domain and range of a relation and its inverse?

The range of the original is the domain of the inverse, and the range of the inverse is the domain of the original.

2. **Discussion** In Step D, you saw that for inverse functions, $f^{-1}(f(x)) = x$. What do you expect $f(f^{-1}(x))$ to equal? Explain.

$\to X$

Explain 1 Finding the Inverse of a Linear Function

Every linear function $f(x) = mx + b$ where $m \neq 0$ is a one-to-one function. So, its inverse is also a function. To find the inverse function, use the fact that inverse functions undo each other's pairings.

To find the inverse of a function $f(x)$:

1. Substitute y for $f(x)$.
2. Solve for x in terms of y.
3. Switch x and y (since the inverse switches inputs and outputs).
4. Replace y with $f^{-1}(x)$. INVERSE

To check your work and verify that the functions are inverses, show that $f(f^{-1}(x)) = x$ and that $f^{-1}(f(x)) = x$.

Example 1 **Find the inverse function $f^{-1}(x)$ for the given function $f(x)$. Use composition to verify that the functions are inverses. Then graph the function and its inverse.**

(A) $f(x) = 3x + 4$

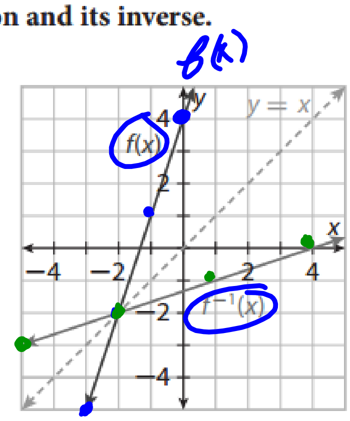
Replace $f(x)$ with y. $\quad y = 3x + 4$

Solve for x. $\quad y - 4 = 3x$

$\frac{y-4}{3} = x$

Interchange x and y. $\quad y = \frac{x-4}{3}$

Replace y with $f^{-1}(x)$. $\quad f^{-1}(x) = \frac{x-4}{3}$



Check: Verify that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.

$$f^{-1}(f(x)) = f^{-1}(3x+4) = \frac{(3x+\cancel{4})\cancel{-4}}{3} = \frac{\cancel{3}x}{\cancel{3}} = x$$

$$f(f^{-1}(x)) = f\left(\frac{x-4}{3}\right) = \cancel{3}\left(\frac{x-4}{\cancel{3}}\right) + 4 = (x - \cancel{4}) + \cancel{4} = x$$

ALWAYS ANSWER

(B) $f(x) = 2x - 2$

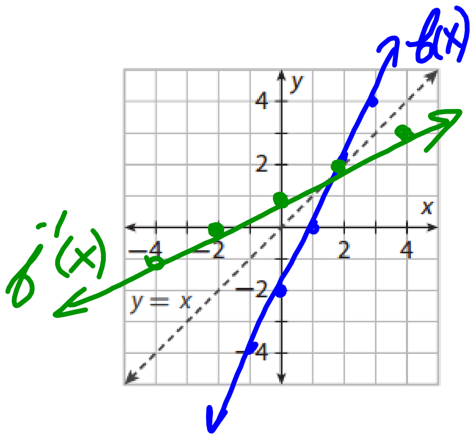
Replace $f(x)$ with y. $\quad y = \boxed{2x-2}$

Solve for x. $\quad y \boxed{+2} = 2x$

$\frac{y+2}{2} = x$

Interchange x and y. $\quad y = \boxed{\frac{x+2}{2}}$

Replace y with $f^{-1}(x)$. $\quad \boxed{f^{-1}(x)} = \frac{x+2}{2}$



Check: Verify that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.

$$f^{-1}(f(x)) = f^{-1}\left(\boxed{2x-2}\right) = \frac{(2x-2)+\boxed{2}}{\boxed{2}} = \frac{\boxed{2x}}{2} = \boxed{x}$$

$$f(f^{-1}(x)) = f\left(\boxed{}\right) = \boxed{2}\left(\frac{x+2}{2}\right) - \boxed{2} = \left(\boxed{x+2}\right) - 2 = \boxed{x}$$

Reflect

3. What is the significance of the point where the graph of a linear function and its inverse intersect?

The lines intersect along y=x. This means the original and inverse are the same at this point

4. The graph of a constant function $f(x) = c$ for any constant c is a horizontal line through the point $(0, c)$. Does a constant function have an inverse? Does it have an inverse function? Explain.

Your Turn

Find the inverse function $f^{-1}(x)$ for the given function $f(x)$. Use composition to verify that the functions are inverses. Then graph the function and its inverse.

5. $f(x) = -2x + 3$

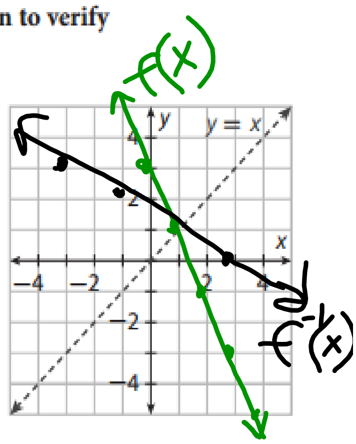
$y = -2x + 3$

$y - 3 = -2x$

$\frac{y-3}{-2} = x$

$\frac{x-3}{-2} = y$

$f^{-1}(x) = \frac{x-3}{-2}$



Verify:

$f^{-1}(f(x)) = f^{-1}(-2x+3) = \frac{-2x+3-3}{-2} = \frac{-2x}{-2} = x$

$f(f^{-1}(x)) = f\left(\frac{x-3}{-2}\right) = -2\left(\frac{x-3}{-2}\right) + 3 = x - 3 + 3 = x$

Explain 2 Modeling with the Inverse of a Linear Function

In a model for a real-world situation, the variables have specific real-world meanings. For example, the distance d (in miles) traveled in time t (in hours) at a constant speed of 60 miles per hour is $d = 60t$. Writing this in function notation as $d(t) = 60t$ emphasizes that this equation describes distance as a function of time.

You can find the inverse function for $d = 60t$ by solving for the independent variable t in terms of the dependent variable d. This gives the equation $t = \frac{d}{60}$. Writing this in function notation as $t(d) = \frac{d}{60}$ emphasizes that this equation describes time as a function of distance. Because the meanings of the variables can't be interchanged, you do not switch them at the end as you would switch x and y when working with purely mathematical functions. As you work with real-world models, you may have to restrict the domain and range.

Example 2 **For the given function, state the domain of the inverse function using set notation. Then find an equation for the inverse function, and graph it. Interpret the meaning of the inverse function.**

(A) The equation $C = 3.5g$ gives the cost C (in dollars) as a function of the number of gallons of gasoline g when the price is \$3.50 per gallon.

The domain of the function $C = 3.5g$ is restricted to nonnegative numbers to make real-world sense, so the range of the function also consists of nonnegative numbers. This means that the domain of the inverse function is $\{C \mid C \geq 0\}$

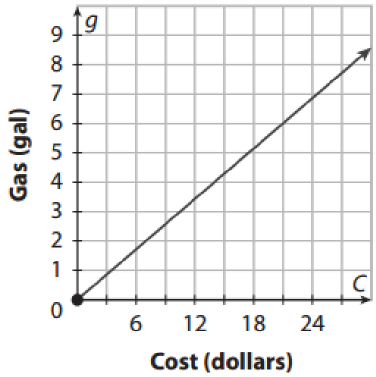
Solve the given equation for g to find the inverse function.

Write the equation. $C = 3.5g$

Divide both sides by 3.5. $\frac{C}{3.5} = g$

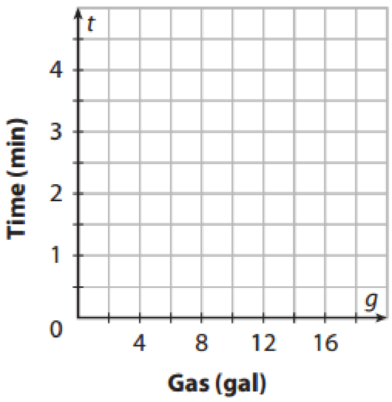
So, the inverse function is $g = \frac{C}{3.5}$.

Graph the inverse function.



The inverse function gives the number of gallons of gasoline as a function of the cost (in dollars) when the price of gas is \$3.50 per gallon.

(B) A car's gas tank, which can hold 14 gallons of gas, contains 4 gallons of gas when the driver stops at a gas station to fill the tank. The gas pump dispenses gas at a rate of 5 gallons per minute. The equation $g = 5t + 4$ gives the number of gallons of gasoline g in the tank as a function of the pumping time t(in minutes).



The range of the function $g = 5t + 4$ is the number of gallons of gas in the tank, which varies from ______ gallons to ______ gallons. So, the domain of the inverse function is $\left\{g \middle| \boxed{} \leq g \leq \boxed{}\right\}$.

Solve the given equation for g to find the inverse function.

Write the equation. $g = \boxed{}\,t + \boxed{}$

Solve for t. $\frac{\boxed{}}{5} = t$

So, the inverse function is $t = \boxed{}$.

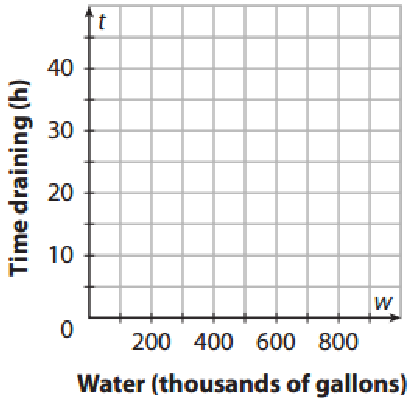
Graph the inverse function.

The inverse function gives ________________________________ as a function of ________________________________.

Your Turn

For the given function, determine the domain of the inverse function. Then find an equation for the inverse function, and graph it. Interpret the meaning of the inverse function.

6. A municipal swimming pool containing 600,000 gallons of water is drained. The amount of water w (in thousands of gallons) remaining in the pool at time t (in hours) after the draining begins is $w = 600 - 20t$.



Elaborate

7. What must be true about a function for its inverse to be a function?

It must be one-to-one

8. A function rule indicates the operations to perform on an input to produce an output. What is the relationship between these operations and the operations indicated by the inverse function?

An inverse undoes these operations. Therefore it performs the inverse operation in reverse order

9. How can you use composition to verify that two functions $f(x)$ and $g(x)$ are inverse functions?

If f(g(x))= x and g(f(x))=x then f(x) and g(x) are inverse functions

10. Describe a real-world situation modeled by a linear function for which it makes sense to find an inverse function. Give an example of how the inverse function might also be useful.

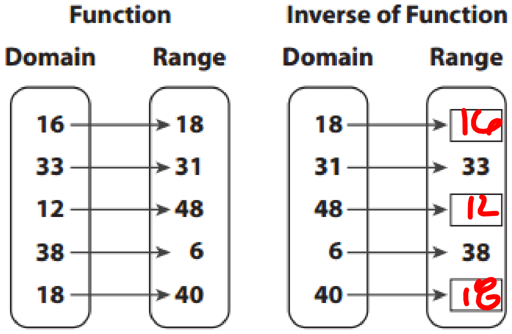
11. **Essential Question Check-In** What is an inverse relation?

Evaluate: Homework and Practice

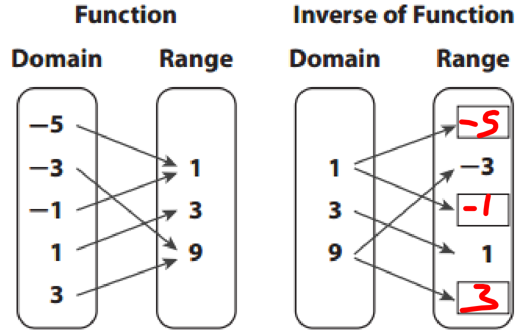
Personal Math Trainer
- Online Homework
- Hints and Help
- Extra Practice

The mapping diagrams show a function and its inverse. Complete the diagram for the inverse of the function. Then tell whether the inverse is a function, and explain your reasoning.

1.



2.

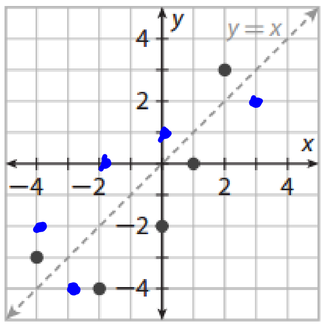


Write the inverse of the given function as a set of ordered pairs and then graph the inverse on the coordinate plane.

3. Function:

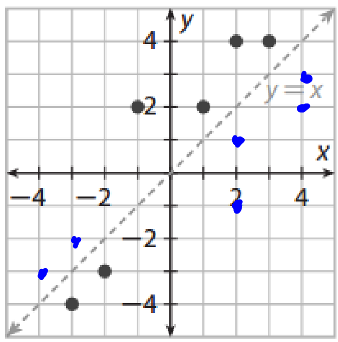
{(−4, −3), (−2, −4), (0, −2), (1, 0), (2, 3)}

{(-3, -4), (-4, -2), (-2, 0), (0, 1) (3, 2)}



4. Function:

{(−3, −4), (−2, −3), (−1, 2), (1, 2), (2, 4), (3, 4)}



Find the inverse function $f^{-1}(x)$ for the given function $f(x)$.

5. $f(x) = 4x - 8$

6. $f(x) = \frac{x}{3}$

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