Real Zeros of Polynomials

Like 3.2, the goal of 3.3 is to find all of the real zeros of a given polynomial.
In section 3.3 we will expand the set of tools we have to accomplish this goal. These tools include

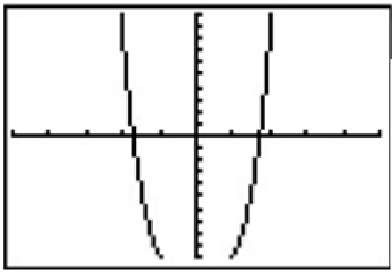
1. The Graphing Calculator
2. Cauchy's Bound
3. Rational Zero's Theorem
4. *u*-substitution

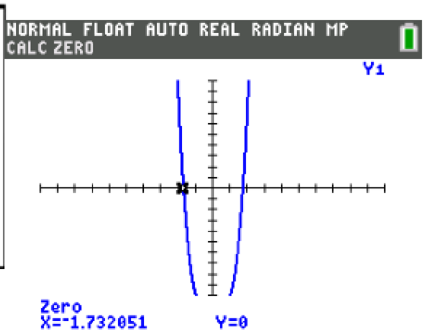
There are several tools given in section 3.3 that you will NOT be responsible for. Those tools are

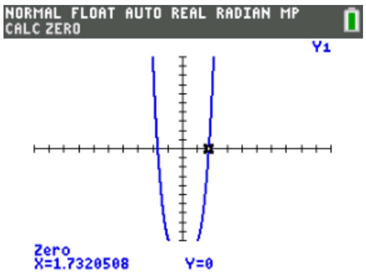
1. Descartes' Rule of Signs
2. Upper and Lower Bounds
3. Bisection Method

Tool 1: Graphing Calculator

Example: Let $f(x) = x^4 + x^2 - 12$. Find all the real zeros of f and their multiplicities.



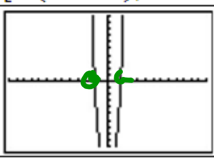




Tool 2: Cauchy's Bound

Theorem 3.8. Cauchy's Bound: Suppose $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ is a polynomial of degree n with $n \geq 1$. Let M be the largest of the numbers: $\frac{|a_0|}{|a_n|}, \frac{|a_1|}{|a_n|}, \ldots, \frac{|a_{n-1}|}{|a_n|}$. Then all the real zeros of f lie in in the interval $[-(M+1), M+1]$.

Example: Let $f(x) = 1x^4 \; +0x^3 \; + 1x^2 \; +0x \; - 12$. Use Cauchy's Bound to determine an interval in which all the real zeros of f lie.

$\left\lvert\frac{a_0}{a_n}\right\rvert, \left\lvert\frac{a_1}{a_n}\right\rvert, \ldots, \left\lvert\frac{a_{n-1}}{a_n}\right\rvert,$	$\left\lvert\frac{a_0}{a_4}\right\rvert = \frac{12}{1}$ $\left\lvert\frac{a_1}{a_4}\right\rvert = \frac{0}{1}$ $\left\lvert\frac{a_2}{a_4}\right\rvert = \frac{1}{1}$ $\left\lvert\frac{a_3}{a_4}\right\rvert = \frac{0}{1}$
M, $M + 1$	$M = 12$ $M + 1 = 13$
Cauchy's Bound	$[-(M-1), M+1] = [-13, 13]$
Graph	

Tool 3: Rational Zero's Theorem

Theorem 3.9. Rational Zeros Theorem: Suppose $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ is a polynomial of degree n with $n \geq 1$, and $a_0, a_1, \ldots a_n$ are integers. If r is a rational zero of f, then r is of the form $\pm\frac{p}{q}$, where p is a factor of the constant term a_0, and q is a factor of the leading coefficient a_n.

Example: Let $f(x) = x^4 + x^2 - 12$. Use the Rational Zeros Theorem to determine a list of possible rational zeros of f.

Factors of a_0 (p)	1, 2, 3, 4, 6, 12
Factors of a_n (q)	1
$\pm\frac{p}{q}$	$\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$

Tool 4: *u*-substitution

Example: Let $f(x) = x^4 + x^2 - 12$. Use u-substitution to find all the real zeros of f and their multiplicities.

~~$x^2 + x - 12$~~
~~$(x+4)(x-3)$~~

Make u-substitution	Let u = x^2
Rewrite f in terms of u	$f(u) = u^2 + u - 12$
Find zeros in terms of u	$f(u) = (u+4)(u-3)$ $0 = (u+4)(u-3)$ $u+4=0$ $u-3=0$ $u=-4$ $u=3$ $u = \frac{-1 \pm \sqrt{1-4(1)(-12)}}{2(1)}$ $u = \frac{-1 \pm 7}{2} = 3, -4$
Find zeros in terms of x	$x^2 = -4$ $x^2 = 3$ $x = \pm\sqrt{-4}$ $x = \pm\sqrt{3}$ $x = \pm 2i$

Note 1: In general, u-substitution can help us identify a `quadratic in disguise' provided that there are exactly three terms and the exponent of the first term is exactly twice that of the second.

Note 2: It is entirely possible that a polynomial has no real roots at all, or worse, it has real roots but none of the techniques discussed in this section can help us find them exactly. In the latter case, we are forced to approximate, which in this section means we use the `Zero' command on the graphing calculator.

Practice

Example:

1. Find all of the real solution to the equation $2x^5 + 6x^3 + 3 = 3x^4 + 8x^2$.

$2x^5 - 3x^4 + 6x^3 - 8x^2 + 3 = 0$

$\pm 1, \pm 3, \pm \frac{1}{2}, \pm \frac{3}{2}$

3 real $(-\frac{1}{2}, 1, 1)$

2 imaginary

$$\begin{array}{r|rrrrrr} 1 & 2x^5 & -3 & 6 & -8 & 0 & 3 \\ & \downarrow & 2 & -1 & 5 & -3 & -3 \\ \hline & 2 & -1 & 5 & -3 & -3 & 0 \end{array}$$

$$\begin{array}{r|rrrrr} 1 & 2x^4 & -1 & 5 & -3 & -3 \\ & \downarrow & 2 & 1 & 6 & 3 \\ \hline & 2 & 1 & 6 & 3 & 0 \end{array}$$

$$\begin{array}{r|rrrr} -\frac{1}{2} & 2x^3 & 1 & 6 & 3 \\ & \downarrow & -1 & 0 & -3 \\ \hline & 2x^2 & 0 & 6 & 0 \end{array}$$

$2x^2 + 6$

$2(x^2+3)$

$2x^2 + 6 = 0$

$2x^2 = -6$

$x^2 = -3$

$x = \pm\sqrt{-3}$

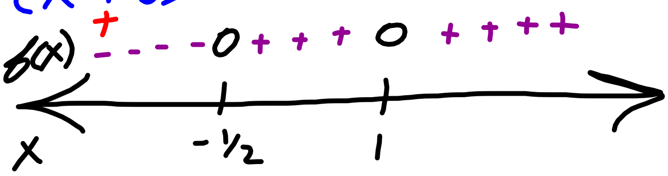
$= \pm i\sqrt{3}$

2. Solve the inequality $2x^5 + 6x^3 + 3 \le 3x^4 + 8x^2$.

$2x^5 - 3x^4 + 6x^3 - 8x^2 + 3 \le 0$

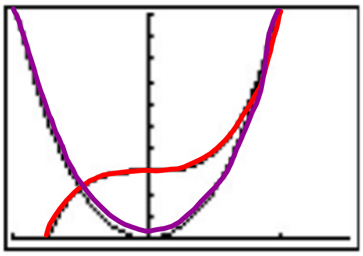
$(x-1)^2 (2x+1)(x^2+3) \le 0$

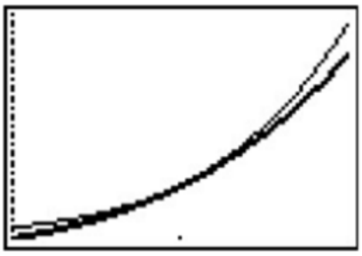
$+$ $+$



Solution: $\left(-\infty, -\frac{1}{2}\right]$

3. Interpret the answer to part 2 graphically, and verify using a graphing calculator.





The pictures on the left show the graph of
$g(x) = 2x^5 + 6x^3 + 3$ and
$h(x) = 3x^4 + 8x^2$

In the second image $h(x) = 3x^4 + 8x^2$ is the thicker line.

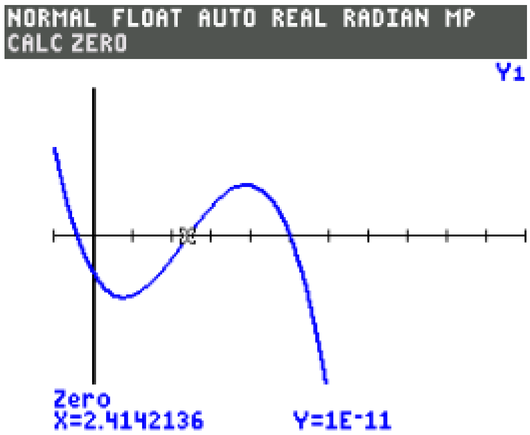
Example: Suppose the profit P, in *thousands* of dollars, from producing and selling *x hundred* LCD TVs is given by $P(x) = -5x^3 + 35x^2 - 45x - 25, 0 \le x \le 10.07$. How many TVs should be produced to make profit?

5	-5	35	-45	-25
		-25	50	25
	-5	10	5	0

$(-5x^2 + 10x + 5)(x-5)$

$-5(x^2 - 2x - 1)(x-5)$

$\frac{2 \pm \sqrt{4+4}}{2} = \frac{2 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2}$



$[242, 499]$

Solution:__________ LCD TVs.

Find a quadratic polynomial with integer coefficients which has $x = \frac{3}{8} \pm \frac{\sqrt{23}}{8}$ as its real zeros.

$$\left(x-\left(\frac{3}{8}+\frac{\sqrt{23}}{8}\right)\right)\left(x-\left(\frac{3}{8}-\frac{\sqrt{23}}{8}\right)\right)$$

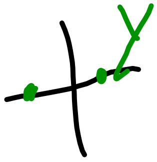
$$\left(x-\frac{3}{8}-\frac{\sqrt{23}}{8}\right)\left(x-\frac{3}{8}+\frac{\sqrt{23}}{8}\right) \qquad \left(x-\frac{3+\sqrt{23}}{8}\right)\left(x-\frac{3-\sqrt{23}}{8}\right)$$

$$x^2-\frac{3}{8}x+\cancel{\frac{\sqrt{23}}{8}x}-\frac{3}{8}x+\frac{9}{64}-\cancel{\frac{3\sqrt{23}}{64}}-\cancel{\frac{\sqrt{23}}{8}x}+\cancel{\frac{3\sqrt{23}}{64}}-\frac{23}{64}$$

$$32\left(x^2-\frac{3}{4}x-\frac{7}{32}\right)$$

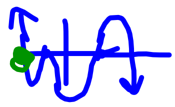
$$32x^2-24x-7$$

$x = 2, -3$



$y = (x-2)(x+3)$

Create a polynomial p with the following attributes.



- As $x \to -\infty$, $p(x) \to \infty$.
- The point $(-1, 0)$ yields a local maximum.
- The degree of p is 5.
- The point $(7, 0)$ is one of the x-intercepts of the graph of p.

$$f(x) = -(x-7)(x+1)^2(x+7)(x\ 0)$$