Pg 175

4.2 Parabolas

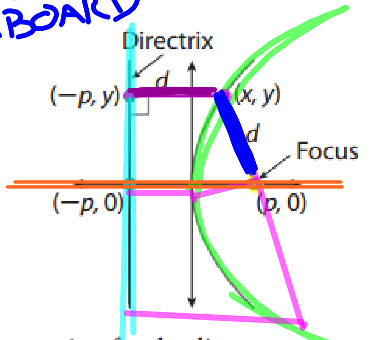
Essential Question: How is the distance formula connected with deriving equations for both vertical and horizontal parabolas?

Resource Locker

Explore Deriving the Standard-Form Equation of a Parabola

BACKBOARD

A parabola is defined as a set of points equidistant from a line (called the directrix) and a point (called the focus). The focus will always lie on the axis of symmetry, and the directrix will always be perpendicular to the axis of symmetry. This definition can be used to derive the equation for a horizontal parabola opening to the right with its vertex at the origin using the distance formula. (The derivations of parabolas opening in other directions will be covered later.)



Ⓐ The coordinates for the focus are given by ☐.

Ⓑ Write down the expression for the distance from a point (x, y) to the coordinates of the focus:

$$d = \sqrt{(x - p)^2 + (y - 0)^2}$$

Ⓒ The distance from a point to a line is measured by drawing a perpendicular line segment from the point to the line. Find the point where a horizontal line from (x, y) intersects the directrix (defined by the line $x = -p$ for a parabola with its vertex on the origin).

☐

Ⓓ Write down the expression for the distance from a point, (x, y) to the point from Step C:

$$d = \sqrt{(x + p)^2 + (y - y)^2}$$

Ⓔ Setting the two distances the same and simplifying gives.

$$\sqrt{(x-p)^2 + y^2} = \sqrt{(x+p)^2}$$

$(x-p)(x-p)$ $(x+p)(x+p)$

To continue solving the problem, square both sides of the equation and expand the squared binomials.

$$1x^2 + -2xp + 1p^2 + y^2 = 1x^2 + 2xp + 1p^2$$

Ⓕ Collect terms.

$$\cancel{x^2} + -4px + \cancel{p^2} + y^2 = 0$$

Ⓖ Finally, simplify and arrange the equation into the **standard form for a horizontal parabola** ~~(with vertex at (0, 0))~~:

$$y^2 = 4px$$

Reflect

1. Why was the directrix placed on the line $x = -p$?

2. **Discussion** How can the result be generalized to arrive at the standard form for a horizontal parabola with a vertex at (h, k):
$(y - k)^2 = 4p(x - h)$?

Explain 1 Writing the Equation of a Parabola with Vertex at (0, 0)

The equation for a horizontal parabola with vertex at $(0, 0)$ is written in the standard form as $y^2 = 4px$. It has a vertical directrix along the line $x = -p$, a horizontal axis of symmetry along the line $y = 0$, and a focus at the point $(p, 0)$. The parabola opens toward the focus, whether it is on the right or left of the origin $(p > 0$ or $p < 0)$. Vertical parabolas are similar, but with horizontal directrices and vertical axes of symmetry:

Parabolas with Vertices at the Origin		
	Vertical	Horizontal
Equation in standard form	$x^2 = 4py$	$y^2 = 4px$
$p > 0$	Opens upward P+	Opens rightward P+
$p < 0$	Opens downward P−	Opens leftward P−
Focus	$(0, p)$	$(p, 0)$
Directrix BACKBOARD	$y = -p$	$x = -p$
Axis of Symmetry	$x = 0$	$y = 0$

Example 1 **Find the equation of the parabola from the description of the focus and directrix. Then make a sketch showing the parabola, the focus, and the directrix.**

(A) Focus $(-8, 0)$, directrix $x = 8$

A vertical directrix means a horizontal parabola.

Confirm that the vertex is at $(0, 0)$:

a. The y-coordinate of the vertex is the same as the focus: 0.

b. The x-coordinate is halfway between the focus (-8) and the directrix $(+8)$: 0.

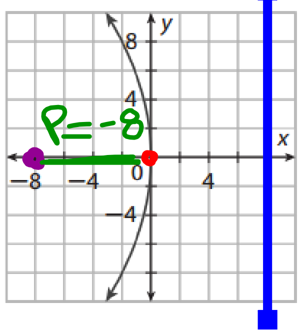
c. The vertex is at $(0, 0)$.

P=-8

Use the equation for a horizontal parabola, $y^2 = 4px$, and replace p with the x coordinate of the focus: $y^2 = 4(-8)x$

Simplify: $y^2 = -32x$

Plot the focus and directrix and sketch the parabola.



Vertex is in the middle of the Focus and Diretrix

P value the distance from focus to vertex

(B) Focus $(0, -2)$, directrix $y = 2$

A [vertical/horizontal] directrix means a [vertical/horizontal] parabola.

Confirm that the vertex is at $(0, 0)$:

a. The x-coordinate of the vertex is the same as the focus: 0.

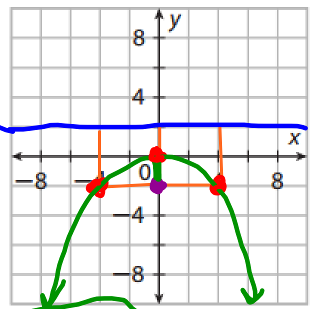
b. The y-coordinate is halfway between the focus, ___ and the directrix, ___ : 0

c. The vertex is at $(0, 0)$.

Use the equation for a vertical parabola, ______, and replace p with the y-coordinate of the focus: $x^2 = 4 \cdot$ ___ $\cdot y$

Simplify: $x^2 =$ ___

Plot the focus, the directrix, and the parabola.



P=-2

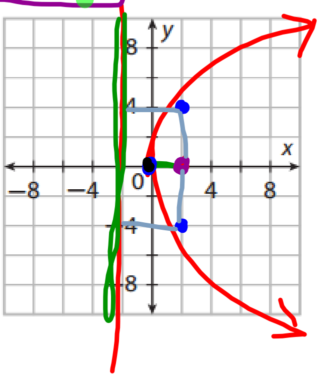
$x^2 = 4Py$

$x^2 = -8y$

Your Turn

Find the equation of the parabola from the description of the focus and directrix. Then make a sketch showing the parabola, the focus, and the directrix.

3. Focus $(2, 0)$, directrix $x = -2$



P=2

$y^2 = 4(2)x$

$y^2 = 8x$

4. Focus $\left(0, -\frac{1}{2}\right)$, directrix $y = \frac{1}{2}$

V: (0,0)

P= $-\frac{1}{2}$

$x^2 = 4Py$

$x^2 = -2y$

Pg 178

Explain 2 Writing the Equation of a Parabola with Vertex at (h, k)

The standard equation for a parabola with a vertex (h, k) can be found by translating from $(0, 0)$ to (h, k): substitute $(x - h)$ for x and $(y - k)$ for y. This also translates the focus and directrix each by the same amount.

Parabolas with Vertex (h, k)		
	Vertical	**Horizontal**
Equation in standard form	$(x - h)^2 = 4p(y - k)$	$(y - k)^2 = 4p(x - h)$
$p > 0$	Opens upward	Opens rightward
$p < 0$	Opens downward	Opens leftward
Focus	$(h, k + p)$	$(h + p, k)$
Directrix	$y = k - p$	$x = h - p$
Axis of Symmetry	$x = h$	$y = k$

p is found halfway from the directrix to the focus:

- For vertical parabolas: $p = \dfrac{(y \text{ value of focus}) - (y \text{ value of directrix})}{2}$
- For horizontal parabolas: $p = \dfrac{(x \text{ value of focus}) - (x \text{ value of directrix})}{2}$

The vertex can be found from the focus by relating the coordinates of the focus to h, k, and p.

Example 2 **Find the equation of the parabola from the description of the focus and directrix. Then make a sketch showing the parabola, the focus, and the directrix.**

h k+p

Ⓐ Focus $(3, 2)$, directrix $y = 0$

V: $(3, 1)$ (h, k)

A horizontal directrix means a vertical parabola.

$p = \dfrac{(y \text{ value of focus}) - (y \text{ value of directrix})}{2} = \dfrac{2 - 0}{2} = 1$

P = 1

h = the x-coordinate of the focus $= 3$

Solve for k: The y-value of the focus is $k + p$, so

$k + p = 2$

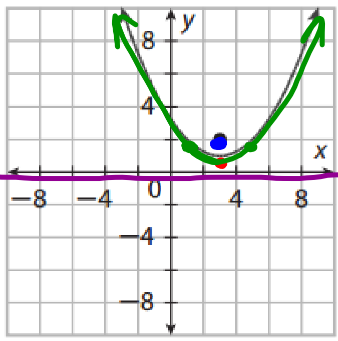
$k + 1 = 2$

$k = 1$

$(x-h)^2 = 4P(y-k)$

Write the equation: $(x - 3)^2 = 4(y - 1)$

$(x-3)^2 = 4(y-1)$



Plot the focus, the directrix, and the parabola.

(B) Focus $(-1, -1)$, directrix $x = 5$

A vertical directrix means a ____________ parabola.

$$p = \frac{(x \text{ value of focus}) - (x \text{ value of directrix})}{2} = \frac{\square - \square}{2} = \square$$

$k =$ the y-coordinate of the focus $= \square$

Solve for h: The x-value of the focus is $h + p$, so

$h + p = \boxed{-1}$

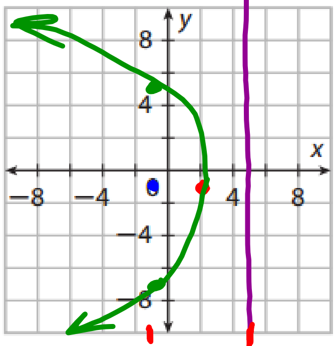
$h + (-3) = \boxed{-1}$

$h = \boxed{2}$

Write the equation: $(y + 1)^2 = \square(x - \square)$

$P = \frac{-1-5}{2} = -3$

$P = -3$



$(y-k)^2 = 4P(x-h)$

$(y+1)^2 = -12(x-2)$

$\frac{-1+5}{2}$

2

$V: (2, -1)$

Your Turn

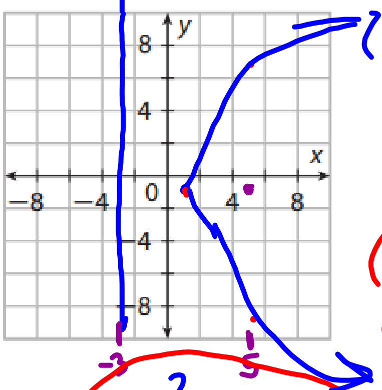
Find the equation of the parabola from the description of the focus and directrix. Then make a sketch showing the parabola, the focus, and the directrix.

5. Focus $(5, -1)$, directrix $x = -3$

$V: (h, k)$

$P = 4$

$P = \frac{5+3}{2}$



$(y+1)^2 = 16(x-1)$

6. Focus $(-2, 0)$, directrix $y = 4$

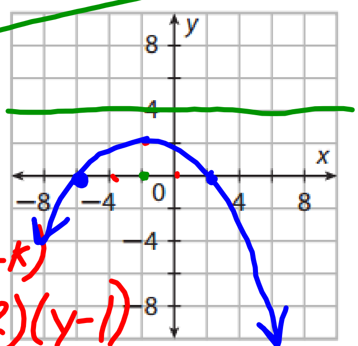
$p = \frac{0-4}{2} = -2$

$v: (-2, 1)$

$(x-h)^2 = 4p(y-k)$

$(x+2)^2 = 4(-2)(y-1)$

$(x+2)^2 = -8(y-1)$



Explain 3 Rewriting the Equation of a Parabola to Graph the Parabola

Pg 183 1-8,15

A **second-degree equation in two variables** is an equation constructed by adding terms in two variables with powers no higher than 2. The general form looks like this:

$$ax^2 + by^2 + cx + dy + e = 0$$

Expanding the standard form of a parabola and grouping like terms results in a second-degree equation with either $a = 0$ or $b = 0$, depending on whether the parabola is vertical or horizontal. To graph an equation in this form requires the opposite conversion, accomplished by completing the square of the squared variable.