Q. 1 a) Let $v_1 = (2, -1, 3, 2)$, $v_2 = (-1, 1, 1, -3)$ and $v_3 = (1, 1, 9, -5)$ be three vectors of the space $\mathbb{R}^4$. Does $(3, -1, 0, -1) \in \text{span}\{v_1, v_2, v_3\}$? Justify your answer.

Sol^n: Let

$$(3, -1, 0, -1) = x(2, -1, 3, 2) + y(-1, 1, 1, -3) + z(1, 1, 9, -5)$$

$$= (2x - y + z,\ -x + y + z,\ 3x + y + 9z,\ 2x - 3y - 5z) \quad - (1)$$

where $x, y, z \in \mathbb{R}$

$$\left.\begin{aligned} 2x - y + z &= 3 \\ -x + y + z &= -1 \\ 3x + y + 9z &= 0 \\ 2x - 3y - 5z &= -1 \end{aligned}\right\} (2)$$

Now $AX = B$

where $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 1 & 1 \\ 3 & 1 & 9 \\ 2 & -3 & -5 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $B = \begin{bmatrix} 3 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

$$[A/B] = \left[\begin{array}{ccc|c} 2 & -1 & 1 & 3 \\ -1 & 1 & 1 & -1 \\ 3 & 1 & 9 & 0 \\ 2 & -3 & -5 & -1 \end{array}\right]$$

$R_1 \leftrightarrow R_2$

$$\Rightarrow [A|B] = \left[\begin{array}{ccc|c} -1 & 1 & 1 & -1 \\ 2 & -1 & 1 & 3 \\ 3 & 1 & 9 & 0 \\ 2 & -3 & -5 & -1 \end{array}\right]$$

$R_1 \to (-1) R_1$

$$\Rightarrow [A|B] = \left[\begin{array}{ccc|c} 1 & -1 & -1 & 1 \\ 2 & -1 & 1 & 3 \\ 3 & 1 & 9 & 0 \\ 2 & -3 & -5 & -1 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$, $R_4 \to R_4 - 2R_1$

$$\Rightarrow [A|B] = \left[\begin{array}{ccc|c} 1 & -1 & -1 & 1 \\ 0 & 1 & 3 & -5 \\ 0 & 4 & 12 & -3 \\ 0 & -1 & -3 & -3 \end{array}\right]$$

$R_3 \to \left(\frac{1}{4}\right) R_3 \quad R_4 \to (R_4 + R_2)$

$$\Rightarrow [A|B] = \left[\begin{array}{ccc|c} 1 & -1 & -1 & 1 \\ 0 & 1 & 3 & -5 \\ 0 & 1 & 3 & -3/4 \\ 0 & 0 & 0 & -8 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$\Rightarrow [A|B] = \left[\begin{array}{ccc|c} 1 & -1 & -1 & 1 \\ 0 & 1 & 3 & -5 \\ 0 & 0 & 0 & 17/4 \\ 0 & 0 & 0 & -8 \end{array}\right]$$

Clearly, $\rho(A) = 2$, $\rho(A|B) = 4$ $\therefore \boxed{\rho(A) < \rho(A|B)}$

$\therefore$ The given equation has no solution.

Hence, $(3, -1, 0, -1) \notin \text{span}\{v_1, v_2, v_3\}$

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Q. 1. b) Find the rank and nullity of the linear transformation:

$T: \mathbb{R}^3 \to \mathbb{R}^3$ given by

$T(x,y,z) = (x+z, x+y+2z, 2x+y+3z)$

Sol^n: Given $T: \mathbb{R}^3 \to \mathbb{R}^3$ and given linear transformation

$T(x,y,z) = (x+z, x+y+2z, 2x+y+3z)$

i) For null space

$$T(x,y,z) = 0$$

$$(x+z, x+y+2z, 2x+y+3z) = 0$$

$$x+z=0 \quad - ①$$
$$x+y+2z=0 \quad - ②$$
$$2x+y+3z=0 \quad - ③$$

from ①, $x = -z$,

② $\Rightarrow$ $x+y-2x=0 \Rightarrow x=y$

Hence we get the null space

$N(T) = \{x(1,1,-1) \mid x \in \mathbb{R}\}$

$\therefore \boxed{\text{Nullity} = 1}$ (as no. of vector in basis of $N(T)$)

ii) By Rank-Nullity theorem,

$$\dim \mathbb{R}^3 = \text{rank}(T) + \text{Nullity}(T)$$

$$3 = \text{rank}(T) + 1$$

$$\therefore \boxed{\text{Rank}(T) = 2}$$

Hence $\boxed{\text{Rank}(T) = 2 \ \& \ \text{Nullity}(T) = 1}$

Q. 1 c) Find the values of p and q for which $\lim_{x\to 0} \frac{x(1+p\cos x)-q\sin x}{x^3}$ exists and equals 1.

Sol^n: $\lim_{x\to 0} \frac{x(1+p\cos x)-q\sin x}{x^3}$ $\left(\frac{0}{0}\right)$ Indeterminate fundamental form

$\therefore$ Applying L' Hospital's rule

$$\lim_{x\to 0} \frac{\frac{d}{dx}\left[x(1+p\cos x)-q\sin x\right]}{\frac{d}{dx}(x^3)} = 1$$

$$\therefore \lim_{x\to 0} \frac{1+p\cos x - px\sin x - q\cos x}{3x^2} = 1$$

$$\therefore \lim_{x\to 0} \frac{1+(p-q)\cos x - xp\sin x}{3x^2} = 1$$

to satisfy limit $1+p-q=0$

$$\Rightarrow \boxed{p+1=q} \quad -(1)$$

Applying L' Hospital's rule

$$\lim_{x\to 0} \frac{-(p-q)\sin x - p\sin x - xp\cos x}{6x} = 1$$

$$\lim_{x\to 0} \frac{(q-p)\sin x - p\sin x - xp\cos x}{6x} = 1$$

$$\lim_{x\to 0} \frac{(q-2p)\sin x - px\cos x}{6x} = 1$$

$$\lim_{x\to 0} \frac{(q-2p)}{6} \cdot \frac{\sin x}{x} - \lim_{x\to 0} \frac{p}{6}\cos x = 1$$

$$\Rightarrow \frac{q-2p}{6} - \frac{p}{6} = 1$$

$$\Rightarrow \quad q - 3p = 6$$

$$q = 3p + 6$$

$$\boxed{q = 3(p+2)} \quad - (2)$$

$$\begin{array}{rl} p - q &= -1 \\ - \quad 3p - q &= -6 \\ \hline -2p &= 5 \end{array}$$

$$q = 1 - 5/2$$

$$\therefore \boxed{p = -\frac{5}{2}} \text{ and } \boxed{q = -\frac{3}{2}}$$

Q. 1 d) Examine the convergence of the integral $\int_0^1 \frac{\log x}{1+x}\, dx$.

Sol^n:

Method 1:

$$f(x) = \frac{\log x}{1+x}\, dx$$

Clearly, f is unbounded at $x=0$

$$I = \int_0^1 \frac{\log x}{1+x} \quad -(1)$$

Only point of infinite discontinuity is $x=0$

$$\therefore \lim_{x\to 0} x^{\mu} f(x) = \lim_{x\to 0} x^{\mu} \frac{\log x}{1+x} = 0, \quad \text{if } \mu>0$$

Hence we choose $0<\mu<1$, then by μ test I is convergent.

Method 2:

Since $\frac{\log x}{1+x}$ is negative on $(0,1]$,

we take $f(x) = \frac{-\log x}{1+x}$

Here '0' is the only point of infinite discontinuity of f on $[0,1]$

take $g(x) = \frac{1}{x^n}$

$$\lim_{x \to 0+} \frac{f(x)}{g(x)} = \lim_{x \to 0+} -\frac{x^n \log x}{1+x} = 0, \text{ if } n>0$$

Taking n between 0 and 1, the integral

$\int_0^1 g(x)\, dx$ is convergent.

$\therefore$ By comparison test,

$\int_0^1 f(x)\, dx$ is convergent.

Hence $\boxed{\int_0^1 \frac{\log x}{1+x}\, dx \text{ is convergent}}$

Q. 1 e) A variable plane which is at a constant distance 3p from the origin O cuts the axes in the points A, B, C respectively. Show that the locus of the centroid of the tetrahedron OABC is

$$9\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right)=\frac{16}{p^2}$$

Sol^n:

Let the equation of the variable plane be

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \quad -①$$

It is given that this plane is at a distance '3p' from (0,0,0)

$$\therefore\ 3p=\frac{1}{\sqrt{\left(\frac{1}{a}\right)^2+\left(\frac{1}{b}\right)^2+\left(\frac{1}{c}\right)^2}}$$

or $$\frac{1}{9p^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} \quad -②$$

Also the plane ① meets the axes in A, B and C. So the co-ordinates O, A, B and C are (0,0,0), (a,0,0), (0,b,0) and (0,0,c) respectively.

Let (x,y,z) be the centroid of the tetrahedron OABC, then

$$x = \frac{1}{4}(0+a+0+0)$$

$$x = \frac{1}{4}a$$

Similarly $y = \frac{1}{4}b$ and $z = \frac{1}{4}c$

or $a = 4x,\ b = 4y,\ c = 4z$

Substituting these values of a, b and c in ②, we have the required locus as

$$\frac{1}{9p^2} = \frac{1}{16x^2} + \frac{1}{16y^2} + \frac{1}{16z^2}$$

or $\boxed{9\left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}\right) = \frac{16}{p^2}}$

Hence proved.

Q. 2 a) If the matrix of a linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ relative to the basis $\{(1,0,0), (0,1,0), (0,0,1)\}$ is $\begin{bmatrix} 1 & 1 & 2 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$

then find the matrix of T relative to the basis $\{(1,1,1), (0,1,1), (0,0,1)\}$

Soln: Let $B = \{(1,0,0), (0,1,0), (0,0,1)\}$

given that $[T]_B = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$

let any $\alpha(x,y,z) \in \mathbb{R}^3$ can be expressed as

$$(x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1)$$

$$T(x,y,z) = x\,T(1,0,0) + y\,T(0,1,0) + z\,T(0,0,1) \quad - (1)$$

$$\left.\begin{aligned} T(1,0,0) &= (1,-1,0) \\ T(0,1,0) &= (1,2,1) \\ T(0,0,1) &= (2,1,3) \end{aligned}\right\} \text{ from } [T]_B \text{ given}$$

$$\therefore (1) \Rightarrow T(x,y,z) = x(1,-1,0) + y(1,2,1) + z(2,1,3)$$

$$\boxed{T(x,y,z) = (x+y+2z,\ -x+2y+z,\ y+3z)}$$

is required transformation

Let $B' = \{(1,1,1), (0,1,1), (0,0,1)\}$

Let $(x,y,z) = a(1,1,1) + b(0,1,1) + c(0,0,1)$

$$(x,y,z) = (a,\ a+b,\ a+b+c)$$

$\therefore \quad a = x$

$a+b = y \quad \Rightarrow b = y - x$

$a+b+c = z \quad \Rightarrow c = z - y$

$$\therefore (x,y,z) = x(1,1,1) + (y-x)(0,1,1) + (z-y)(0,0,1)$$

We need to express with respect to B'

$T(1,1,1) = (4,2,4) = 4(1,1,1) - 2(0,1,1) + 2(0,0,1)$

$T(0,1,1) = (3,3,4) = 3(1,1,1) + 0(0,1,1) + 1(0,0,1)$

$T(0,0,1) = (2,1,3) = 2(1,1,1) - 1(0,1,1) + 2(0,0,1)$

Hence matrix of linear transformation is

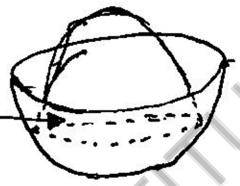
$$[T]_{B'} = \begin{bmatrix} 4 & 3 & 2 \\ -2 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix}$$

is required solution.

Q. 2. b) Evaluate the triple integral which gives the volume of the solid enclosed between the two paraboloids $z = 5(x^2+y^2)$ and $z = 6-7x^2-y^2$.

Soln:

To find volume of solid enclosed betn these two paraboloids



The bounded region of the above paraboloids is

$$z = 5x^2+5y^2 = 6-7x^2-y^2$$

$$12x^2+6y^2 = 6$$

$$2x^2+y^2 = 1$$

is intersection of two curves.

Thus volume of region

$$V = \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \int_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} \int_{5x^2+5y^2}^{6-7x^2-y^2} 1 \, dx\, dy\, dz \quad -(1)$$

Proceeding with cylindrical co-ordinate

$$x = \frac{r}{\sqrt{2}}\cos\theta \quad y = r\sin\theta \quad z = z$$

$$dx\,dy\,dz = \frac{r}{\sqrt{2}}\, dr\, d\theta\, dz$$

On reducing the equation (1)

$$V = \int_0^{2\pi}\int_0^{1}\int_{\frac{5}{2}r^2\cos^2\theta + 5r^2\sin^2\theta}^{6-\frac{7}{2}r^2\cos^2\theta - r^2\sin^2\theta} 1 \cdot \frac{r}{\sqrt{2}}\, dr\, d\theta\, dz$$

$$= \frac{1}{\sqrt{2}}\int_0^{2\pi}\int_0^{1}(6-6r^2)\, r\, dr\, d\theta$$

$$= \frac{1}{\sqrt{2}}\, 2\pi \int_0^1 (6-6r^2)\, r\, dr$$

$$= \frac{1}{\sqrt{2}}\, 2\pi \left(\frac{6r^2}{2} - \frac{6r^4}{4}\right)_0^1$$

$$= \frac{2\pi}{\sqrt{2}}\left(3 - \frac{3}{2}\right)$$

$$= \frac{2\pi}{\sqrt{2}} \cdot \frac{3}{2}$$

$$= \frac{3\pi}{\sqrt{2}}$$

$$\therefore \boxed{V = \frac{3\pi}{\sqrt{2}}}$$

Q. 2 c) i) Show that the equation ~~$2x^2$~~ $2x^2+3y^2-8x+6y-12z+11=0$ represents an elliptic paraboloid. Also find its principal axis and principal planes.

Sol^n: $2x^2+3y^2-8x+6y-12z+11=0$

$a=2 \quad b=3 \quad c=0$

$f=0 \quad g=0 \quad h=0$

$u=-4 \quad v=3 \quad w=-6 \quad d=11$

We know that the discriminating cubic equation is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = 0$$

$$\lambda(\lambda-3)(\lambda-2)=0$$

$$\lambda=0, \quad \lambda=2, \quad \lambda=3$$

Now putting $\lambda=0$ in the determinant given by ① and associating each row with l_3, m_3, n_3 we have

$l_3 = 0, \quad m_3 = 0, \quad n_3 \cdot 0 = 0 \quad n_3 \neq 0$

because $l^2+m^2+n^2 = 1$

$\therefore \ n = 1$

Now $k = u l_3 + v m_3 + w n_3$

$k = -4\times 0 + 3\times 0 + -6\times 1$

$\boxed{k = -6}$

Required reduced eq^n is

$\lambda_1 x^2 + \lambda_2 y^2 + 2kz = 0$

$2x^2 + 3y^2 - 12z = 0$

$$\boxed{\frac{x^2}{3} + \frac{y^2}{2} = 2z}$$

which represent an elliptical paraboloid as both λ_1 and λ_2 are positive

Also if $F(x,y,z) = 0$ be the given surface then the co-ordinate of its vortex are given by solving any two of these equations.

$$\frac{\partial F/\partial x}{l_3} = \frac{\partial F/\partial y}{m_3} = \frac{\partial F/\partial z}{n_3} = 2k$$

and $k(l_3 x + m_3 y + n_3 z) + ux + vy + wz + d = 0$

$$F \equiv 2x^2 + 3y^2 - 8x + 6y - 12z + 11 = 0$$

$$\frac{\partial F}{\partial x} = 4x - 8 \; ; \; \frac{\partial F}{\partial y} = 6y + 6 \; ; \; \frac{\partial F}{\partial z} = -12$$

$$k(l_3 x + m_3 y + n_3 z) + ux + vy + wz + d = 0$$

$$-12(0 \cdot x + 0 \cdot y + 1 \cdot z) - 4x + 3y - 6z + 11 = 0$$

$$-12z - 4x + 3y - 6z + 11 = 0$$

$$-4x + 3y - 18z + 11 = 0$$

$$\frac{\frac{\partial F}{\partial x}}{l_3} = \frac{\frac{\partial F}{\partial y}}{m_3} = \frac{\frac{\partial F}{\partial z}}{n_3} = 2k$$

$$\frac{4x-8}{0} = \frac{6y+6}{0} = \frac{-12}{1} = 2(-12)$$

$\Rightarrow 4x - 8 = 0 \Rightarrow x = 2$ | $6y + 6 = 0$, $y = -1$ | $-8 - 3 - 18z + 11 = 0$, $-18z = 0$, $\Rightarrow z = 0$

$\therefore$ Vertex is $(2, -1, 0)$

$$\boxed{\text{Principal axis equation is } \quad \frac{x-2}{0} = \frac{y+1}{0} = \frac{z-0}{1}}$$

Principal plane equation is given by

$$\lambda(lx + my + nz) + (ul + vm + wn) = 0$$

for $\underline{\lambda=2}$:

$$\begin{vmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{vmatrix}\begin{pmatrix} l_1 \\ m_1 \\ n_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$m_1 = 0 ; \quad n_1 = 0 \Rightarrow l_1 = 1$$

$$\lambda (l_1 x + m_1 y + n_1 z) + (u l_1 + v m_1 + w n_1) = 0$$

$$\therefore \quad 2(x) + (-4 \times 1) = 0$$

$$\Rightarrow \boxed{x - 2 = 0}$$

for $\underline{\lambda = 3}$:

$$\begin{vmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{vmatrix}\begin{pmatrix} l_2 \\ m_2 \\ n_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow -l_2 = 0 ; \quad n_2 = 0 \quad \Rightarrow m_2 = 1$$

$$\lambda (l_2 x + m_2 y + n_2 z) + (u l_2 + v m_2 + w n_2) = 0$$

$$\therefore \quad 3y + 3 = 0$$

$$\Rightarrow \boxed{y + 1 = 0}$$

$\therefore$ Required principal plane is

$$\boxed{\begin{array}{l} x - 2 = 0 \ \& \\ y + 1 = 0 \end{array}}$$

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Q. 2 c) ii) The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ meets the co-ordinate axes in A, B, C respectively. Prove that the equation of the cone generated by the lines drawn from the origin O to meet the circle ABC is

$$yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{b}{a}+\frac{a}{b}\right)=0$$

Sol^n:

The plane ABC is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ — ①

It meets the axes at $A(a,0,0)$, $B(0,b,0)$ & $C(0,0,c)$.

Since the sphere intercepts a length 'a' on x-axis so it passes through the point $(a,0,0)$. Similarly it passes through the points $(0,b,0)$ & $(0,0,c)$. Also it passes through the origin i.e. $(0,0,0)$

Let the equation of sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad — ②$$

If it passes through $(0,0,0)$ then from ② we have $d=0$ — ③

If ② passes through $(a,0,0)$, then we get $a^2+2ua+d=0$

or from ③, $a^2+2ua+0=0$

or $u = -\frac{1}{2}a$, as $a \neq 0$

Similarly as ② passes through $(0, b, 0)$ and $(0, 0, c)$ we get

$$v = -\frac{1}{2}b \quad \text{and} \quad w = -\frac{1}{2}c$$

Hence from ②, required equation is

$$x^2+y^2+z^2-ax-by-cz=0 \quad -④$$

The required cone is generated by the lines drawn from O to meet the circle ABC (given by ① and ④ together) and will be homogeneous. So making ④ homogeneous with the help of ①, we get the required equation as

$$x^2+y^2+z^2-(ax+by+cz)\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)=0$$

or $\boxed{yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{b}{a}+\frac{a}{b}\right)=0}$

Hence proved.

Q. 3 a) Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

i) Verify the Cayley-Hamilton theorem for the matrix A.

ii) Show that $A^n = A^{n-2} + A^2 - I$ for $n \geq 3$, where I is the identity matrix of order 3. Hence find A^{40}.

Sol^n: i) Characteristic (Ch.) equation of A is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(\lambda^2 - 1) = 0$$

$$\lambda^3 - \lambda^2 - \lambda + 1 = 0$$

Ch. eq^n of A is $\lambda^3 - \lambda^2 - \lambda + 1 = 0$

By Cayley-Hamilton theorem

$$A^3 - A^2 - A + I = 0 \quad - (1)$$

L.H.S. $= A^3 - A^2 - A + I$

$$A^2 = A \cdot A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\therefore L.H.S. = A^3 - A^2 - A + I$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= R.H.S.$$

Hence Cayley-Hamilton theorem is verified.

ii) By Principle of mathematical induction

① $n = 3$ $\quad A^3 = A + A^2 - I$ is true by eqⁿ ①

② $n = k$ $\quad A^k = A^{k-2} + A^2 - I$ is assume to be true

③ $n = k+1$ $\quad A^{k+1} = A^{k-1} + A^2 - I$

$$L.H.S. = A^{k+1} = A^k \cdot A$$

$$= (A^{k-2} + A^2 - I)A$$

$$= (A^{k-1} + A^3 - A)$$

$$= A^{k-1} + A^2 + A - I - A$$

$$= A^{k-1} + A^2 - I$$

$$= R.H.S.$$

Hence we proved

$$\boxed{A^n = A^{n-2} + A^2 - I} \text{ for } n \geq 3$$

$n = 4$

$$A^4 = 2A^2 - I$$

$$A^8 = 4A^4 - 4A^2 + I$$

$$= 4(2A^2 - I) - 4A^2 + I$$

$$= 4A^2 - 3I$$

$$A^{16} = 16A^4 - 24A^2 + 9I$$

$$= 16(2A^2 - I) - 24A^2 + 9I$$

$$= 8A^2 - 7I$$

$A^4 = 2A^2 - I$

$A^8 = 4A^2 - 3I$

$A^{16} = 8A^2 - 7I$

$A^{40} = 20A^2 - 19I$

To find A^{40} :-

$$A^{40} = A^{38} + A^2 - I$$

$$= A^{36} + 2A^2 - 2I$$

$$= A^{34} + 3A^2 - 3I$$

$$\vdots$$

$$= A^2 + 19A^2 - 19I$$

$$A^{40} = 20A^2 - 19I$$

$$= \begin{bmatrix} 20 & 0 & 0 \\ 20 & 20 & 0 \\ 20 & 0 & 20 \end{bmatrix} - \begin{bmatrix} 19 & 0 & 0 \\ 0 & 19 & 0 \\ 0 & 0 & 19 \end{bmatrix}$$

$$\therefore \boxed{A^{40} = \begin{bmatrix} 1 & 0 & 0 \\ 20 & 1 & 0 \\ 20 & 0 & 1 \end{bmatrix}}$$ is required solution

Q. 3. b) Justify whether $(0,0)$ is an extreme point for the function $f(x,y) = 2x^4 - 3x^2y + y^2$

Sol^n: Given $f(x,y) = 2x^4 - 3x^2y + y^2$

$$p = \frac{\partial f}{\partial x} = 8x^3 - 6xy \qquad r = \frac{\partial^2 f}{\partial x^2} = 24x^2 - 6y$$

$$q = \frac{\partial f}{\partial y} = -3x^2 + 2y \qquad t = \frac{\partial^2 f}{\partial y^2} = 2$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = -6x$$

At point $(0,0)$:

$$\left(\frac{\partial f}{\partial x}\right)_{(0,0)} = 8(0)^3 - 6(0)(0) = 0$$

$$\left(\frac{\partial f}{\partial y}\right)_{(0,0)} = -3(0)^2 + 2(0) = 0$$

$$\left(\frac{\partial^2 f}{\partial x^2}\right)_{(0,0)} = 24(0)^2 - 6(0) = 0$$

$$\left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(0,0)} = -6(0) = 0$$

$$\left(\frac{\partial^2 f}{\partial y^2}\right)_{(0,0)} = 2$$

Now, $rt - s^2 = 0 \times 2 - (0)^2 = 0$

and $r = 0$ at $(0,0)$

$\therefore$ $(0,0)$ does not meet the condition for extreme point using second partial derivative.

It is a critical point but not an extreme point for the given function.

Hence above function neither have maxima nor minima.

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Q. 3 c) Find the equation of the sphere through the circle $x^2+y^2+z^2-4x-6y+2z-16=0$; $3x+y+3z-4=0$ in the following cases.

i) the point $(1,0,-3)$ lies on the sphere.

ii) the given circle is a great circle of the sphere.

Soln:

We know that, equation of the sphere passing through the circle $S=0=P$ is

$$S+\lambda P=0$$

$$\Rightarrow x^2+y^2+z^2-4x-6y+2z-16+\lambda(3x+y+3z-4)=0$$

$$\Rightarrow x^2+y^2+z^2+(3\lambda-4)x+(\lambda-6)y+(3\lambda+2)z$$
$$-4\lambda-16=0 \quad -\ 1$$

i) But given that above sphere passes through the point $(1,0,-3)$

$$\Rightarrow (1+9-4-6-16)+\lambda(3-9-4)=0$$

$$-16+\lambda(-10)=0$$

$$\lambda=\frac{-16}{10}=\frac{-8}{5}$$

$\therefore$ Sphere equation

$$x^2+y^2+z^2-4x-6y+2z-16-\frac{8}{5}(3x+y+3z-4)=0$$

$$5x^2+5y^2+5z^2-44x-38y-40z-48=0$$

$$\boxed{x^2+y^2+z^2-\frac{44}{5}x-\frac{38}{5}y-8z-\frac{48}{5}=0}$$

Q. 4. a) Find the rank of the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ -1 & 3 & 0 & -4 \\ 2 & 1 & 3 & -2 \\ 1 & 1 & 1 & -1 \end{bmatrix}$$

by reducing it to row-reduced echelon form.

Sol^n:

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ -1 & 3 & 0 & -4 \\ 2 & 1 & 3 & -2 \\ 1 & 1 & 1 & -1 \end{bmatrix}$$

$R_2 \to R_2 + R_1$, $R_3 \to R_3 - 2R_1$, $R_4 \to R_4 - R_1$

$$A \sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 5 & -1 & -4 \\ 0 & -3 & 5 & -2 \\ 0 & -1 & 2 & -1 \end{bmatrix}$$

$R_2 \longleftrightarrow R_4$

$$A \sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & -3 & 5 & -2 \\ 0 & 5 & -1 & -4 \end{bmatrix}$$

$R_2 \to (-1) R_2$

$$A \sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & -3 & 5 & -2 \\ 0 & 5 & -1 & -4 \end{bmatrix}$$

$R_3 \to R_3 + 3R_2$, $R_4 \to R_4 - 5R_2$

$$A \sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 9 & -9 \end{bmatrix}$$

$R_3 \to (-1)$

$$A \sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 9 & -9 \end{bmatrix}$$

$R_4 \to R_4 - 9R_3$

$$A \sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Above matrix is in echelon form. Hence rank of matrix A is non-zero row in echelon form

$$\boxed{\rho(A) = 3}$$

Q. 4 b) Trace the curve $y^2(x^2-1) = 2x-1$

Sol^n: Steps :-

1. Curve is symmetrical about x-axis
2. It does not pass through the origin.
3. Curve meets x-axis in $\left(\frac{1}{2}, 0\right)$ and Y-axis in $(0,1)$ and $(0,-1)$

   Line $x = \frac{1}{2}$ is tangent to curve at $\left(\frac{1}{2}, 0\right)$
4. $x = \pm 1$, $y = 0$ are its only asymptote
5. $y^2 = \dfrac{2x-1}{x^2-1}$

   $\Rightarrow$ for $x < -1$ and $\frac{1}{2} < x < 1$, y is imaginary

   $\therefore$ Curve lies between region $-1 < x < \frac{1}{2}$ & $x > 1$
6. $y = \pm\sqrt{\dfrac{2x-1}{x^2-1}}$

   $\Rightarrow \dfrac{dy}{dx} = \pm\left(\dfrac{-x^2+x+1}{(2x-1)^{1/2}(x^2-1)^{3/2}}\right)$

   for $x > 1$, $\dfrac{dy}{dx} < 0$

   $\Rightarrow y$ is decreasing for $x > 1$

   for $-1 < x < \frac{1}{2}$, $\dfrac{dy}{dx} < 0$

   $\Rightarrow y$ is decreasing in $\left[-1, \frac{1}{2}\right]$

7.

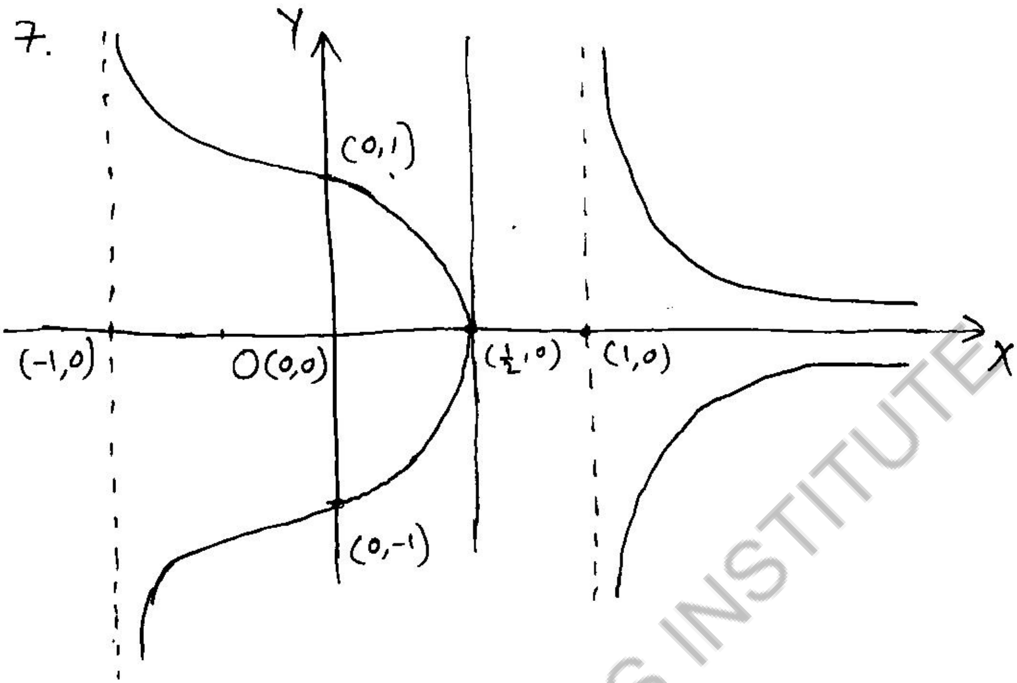


fig. Curve $y^2(x^2-1) = 2x-1$

Q. 4. c) Prove that the locus of a line which meets the lines $y = mx,\ z = c$;
$y = -mx,\ z = -c$ and the circle $x^2+y^2=a^2$, $z=0$ is

$$c^2m^2(cy-mzx)^2 + c^2(yz-cmx)^2 = a^2m^2(z^2-c^2)^2$$

Sol^n: The given lines are

$$y - mx = 0,\quad z - c = 0 \quad -①$$

$$y + mx = 0,\quad z + c = 0 \quad -②$$

and the circle is

$$x^2+y^2=a^2;\quad z=0 \quad -③$$

Any line intersecting ① & ② is

$$\left.\begin{aligned} y - mx + k_1(z-c) &= 0 \\ y + mx + k_2(z+c) &= 0 \end{aligned}\right\} \quad -④$$

If it meets the circle ③, we have to eliminate x, y, z from ③ & ④

putting $z=0$ in ④, we get

$$y - mx + k_1(-c) = 0$$

$$y + mx + k_2(c) = 0$$

Solving $\dfrac{y}{-mk_2c + mk_1c} = \dfrac{x}{-ck_1 - ck_2} = \dfrac{1}{m+m}$

$$\Rightarrow \quad x = \frac{-(k_1+k_2)c}{2m}$$

$$y = \frac{c(k_1-k_2)}{2}$$

putting these values of x, y in ③, we get

$$\frac{c^2(k_1+k_2)^2}{4m^2} + \frac{c^2(k_1-k_2)^2}{4} = a^2$$

$$\Rightarrow \quad c^2(k_1+k_2)^2 + c^2m^2(k_1-k_2)^2 = 4a^2m^2 \quad \text{— ⑤}$$

To find the locus,

eliminate k_1, k_2 from ④ & ⑤

$$\therefore ④ \equiv k_1 = \frac{-(y-mx)}{z-c} = \frac{mx-y}{z-c}$$

$$k_2 = \frac{-(y+mx)}{z+c}$$

substituting these values in ⑤

$$\therefore ⑤ \Rightarrow c^2\left[\left(\frac{mx-y}{z-c}\right) + \left(\frac{-mx-y}{z+c}\right)\right]$$

$$+ c^2m^2\left[\left(\frac{mx-y}{z-c}\right) + \left(\frac{mx+y}{z+c}\right)\right] = 4a^2m^2$$

on simplification we get

$$c^2m^2(cy-mzx)^2 + c^2(yz-cmx)^2 = a^2m^2(z^2-c^2)^2$$

which is the required locus.

Q 5 a) obtain the solution of the initial value problem $\frac{dy}{dx} - 2xy = 2$, $y(0)=1$ in the form $y = e^{x^2}[1+\sqrt{\pi}\,\mathrm{erf}(x)]$.

soln — $\frac{dy}{dx} - 2xy = 2$ is Linear differential eqn in y.

$P = -2x \qquad q = 2$

$$IF = e^{\int P\,dx} = e^{\int -2x\,dx} = e^{-x^2}$$

Hence solution is

$$y \cdot IF = \int q\, IF\, dx + c$$

$$y \cdot e^{-x^2} = \int 2e^{-x^2} dx + c \quad \text{—— (1)}$$

$y(0) = 1$

$$\therefore 1 \cdot e^{0} = \int 2e^{0} \cdot 0 + c$$

$$\boxed{1 = c}$$

eqn (1) become

$$y \cdot e^{-x^2} = \int 2e^{-x^2} dx + 1$$

$$y = e^{x^2}\left[\int 2e^{-x^2} dx + 1\right] \quad \text{—— (2)}$$

$$\int e^{-x^2} dx = \int_0^x e^{-u^2} du \quad \text{—— (3)}$$

we know $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du$

$$\therefore\ 2\int_0^x e^{-u^2}\,du = \sqrt{\pi}\ \operatorname{erf}(x) \quad \text{——} \ (4)$$

hence by eq^n (2), (3) and (4)

$$\boxed{y = e^{x^2}\left[1 + \sqrt{\pi}\ \operatorname{erf}(x)\right]}$$

is required solution.

Q. 5 b) Given that $L\{f(t); p\} = F(p)$

Show that $\int_0^\infty \frac{f(t)}{t}\,dt = \int_0^\infty F(x)\,dx.$

Hence evaluate the integral $\int_0^\infty \frac{e^{-t} - e^{-3t}}{t}\,dt.$

Sol^n: i) Given that $L\{f(t); p\} = F(p)$

We have,

$$L\left\{\frac{f(t)}{t}\right\} = \int_0^\infty e^{-st}\frac{f(t)}{t}\,dt$$

$$= \int_s^\infty f(s)\,ds$$

$$= \int_s^\infty F(x)\,dx$$

$$= \int_s^0 F(x)\,dx + \int_0^\infty F(x)\,dx$$

$$= -\int_0^s F(x)\,dx + \int_0^\infty F(x)\,dx \quad -(1)$$

Taking limits of both sides of (1) as $s \to 0+$ and assuming that the integral converges, we get

$$\boxed{\int_0^\infty \frac{f(t)}{t}\,dt = \int_0^\infty F(x)\,dx}$$

Hence proved.

ii) $\displaystyle\int_0^\infty \frac{e^{-t}-e^{-3t}}{t}\,dt$

$\Rightarrow L\{e^{-t}-e^{-3t}\}$

$= L\{e^{-t}\} - L\{e^{-3t}\}$

$= \dfrac{1}{s+1} - \dfrac{1}{s+3} = f(s)$, say $\quad -(1)$

$$L\left\{\frac{e^{-t}-e^{-3t}}{t}\right\} = \int_s^\infty f(s)\,ds$$

$= \displaystyle\int_s^\infty \left(\frac{1}{s+1} - \frac{1}{s+3}\right) ds$, using (1)

$= \left[\log(s+1) - \log(s+3)\right]_s^\infty$

$= \left[\log\dfrac{(s+1)}{(s+3)}\right]_s^\infty$

$= \lim\limits_{s\to\infty} \log\dfrac{s+1}{s+3} - \log\left(\dfrac{s+1}{s+3}\right)$

$= \lim\limits_{s\to\infty} \log\dfrac{1+1/s}{1+3/s} + \log\left(\dfrac{s+3}{s+1}\right)$

$= \log(1) + \log\left(\dfrac{s+3}{s+1}\right)$

$= \log\left(\dfrac{s+3}{s+1}\right)$

or $\displaystyle\int_0^\infty e^{-st}\left(\frac{e^{-t}-e^{-3t}}{t}\right) dt = \log\left(\frac{s+3}{s+1}\right) \quad -(2)$

– by definition of Laplace Transform

Taking limit of both sides of ②

as $s \to 0$, we get

$$\boxed{\int_0^\infty \frac{e^{-t} - e^{-3t}}{t}\, dt = \log\left(\frac{3}{1}\right) = \log 3}$$

Q.5c) A cylinder of radius 'a' touches a vertical wall along a generating line. Axis of the cylinder is fixed horizontally. A uniform flat beam of length 'l' and weight 'W' rests with its extremities in contact with the wall and the cylinder, making an angle of 45° with the vertical. If frictional forces are neglected then show that

$$\frac{a}{l} = \frac{\sqrt{5}+5}{4\sqrt{2}}$$

Also, find the reactions of the cylinder and wall.

sol^n ⇒

A Given the rod AB is in the position of ~~equation~~ equilibrium on the cylinder and against the wall under the forces then it must be true below.

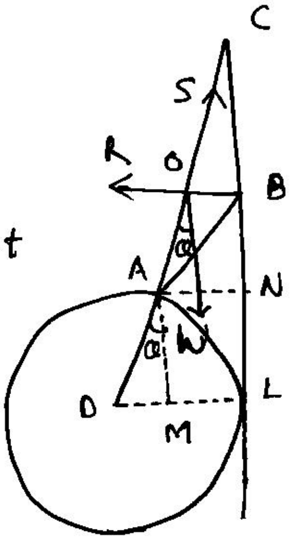


fig. 01

i) Reaction R ⊥ wall at B

ii) Reaction S at A passing through the centre D.

iii) weight W acting at the middle point G of the rod vertically downwards

$\angle ABL = \angle OGB = 45^\circ$ (Given)

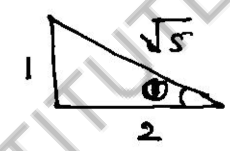
Let $\angle AOG = \theta$

Applying m:n theorem in ΔAOB

$(1+1)\cot 45^\circ = 1\cot\theta - 1\cot 90^\circ$

$2 = \cot\theta$

$\cot\theta = 2$ —— (1)



by applying Lami's theorem

$$\frac{R}{\sin(180-\theta)} = \frac{S}{\sin 90} = \frac{W}{\sin(90+\theta)}$$

$$\frac{R}{\sin\theta} = \frac{S}{1} = \frac{W}{\cos\theta}$$

$R = W\tan\theta = \frac{W}{2}$ since eq^n (1)

Reaction on the wall $R = \frac{W}{2}$

and $S = W\sec\theta = W\sqrt{1+\tan^2\theta}$

$= W\sqrt{1+1/4} = \frac{\sqrt{5}}{2}W$

$DL = DM + ML = DM + AN = a\sin\theta + l\sin 45^\circ$

$\Rightarrow a = a\left(\frac{1}{\sqrt{5}}\right) + l\left(\frac{1}{\sqrt{2}}\right)$

$\Rightarrow \frac{l}{\sqrt{2}} = \left(\frac{\sqrt{5}-1}{\sqrt{5}}\right)a$

$$\Rightarrow \frac{a}{l} = \frac{\sqrt{5}\cdot 1}{\sqrt{2}\,\sqrt{5-1}} = \frac{\sqrt{5}}{\sqrt{2}}\cdot\frac{\sqrt{5+1}}{\sqrt{5-1}\,\sqrt{5+1}}$$

$$= \frac{\sqrt{5}}{\sqrt{2}}\cdot\frac{\sqrt{5}+1}{4} = \frac{\sqrt{5}+5}{4\sqrt{2}}$$

$$\therefore \boxed{\frac{a}{l} = \frac{5+\sqrt{5}}{4\sqrt{2}}}$$ Hence proved.

similar, Reaction on the wall

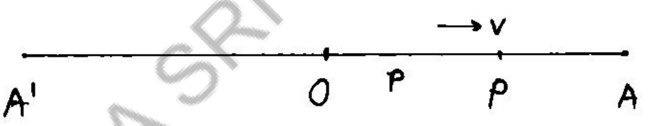
$$R = w\tan\alpha = w\cdot\left(\frac{1}{2}\right) = \frac{w}{2}$$

$$\therefore \boxed{R = \frac{w}{2}}$$

is required solution.

Q. 5 d) A particle is moving under Simple Harmonic Motion of period T about a centre O. It passes through the point P with velocity v along the direction OP and $OP = p$. Find the time that elapses before the particle returns to the point P. What will be the value of p when the elapsed time is $\frac{T}{2}$?

Sol^n: Let the equation of the S.H.M. with centre O as origin be $\frac{d^2x}{dt^2} = -\mu x$



The time period $T = \frac{2\pi}{\sqrt{\mu}}$

Let the amplitude be a.

Then $\left(\frac{dx}{dt}\right)^2 = \mu(a^2 - x^2)$ — ①

When the particle passes through P its velocity is given to be 'v' in the direction OP. Also $OP = p$. So putting $x = p$ and $\frac{dx}{dt} = v$ in ①, we get

$$v^2 = \mu(a^2 - p^2). \quad - ②$$

Let A be an extremity of the motion. From P the particle comes to instantaneous rest at A and then returns back to P.

In S.H.M. the time from P to A is equal to the time from A to P.

$\therefore$ the required time $= 2 \times$ (time from A to P)

Now for the ~~at~~ motion from A to P, we have

$$\frac{dx}{dt} = -\sqrt{\mu(a^2-x^2)} \quad \text{or} \quad dt = -\frac{dx}{\sqrt{\mu(a^2-x^2)}} \quad \text{--- (3)}$$

Let t_1 be the time from A to P.

Then at A, $t=0$, $x=a$ and at P, $t=t_1$ & $x=p$.

Therefore integrating (3), we get

$$\int_0^{t_1} dt = \frac{1}{\sqrt{\mu}} \int_a^p \frac{-dx}{\sqrt{a^2-x^2}}$$

$$t_1 = \frac{1}{\sqrt{\mu}}\left[\cos^{-1}\frac{x}{a}\right]_a^p$$

$$\therefore \quad t_1 = \frac{1}{\sqrt{\mu}}\left[\cos^{-1}\frac{p}{a} - \cos^{-1} 1\right]$$

$$\therefore \quad t_1 = \frac{1}{\sqrt{\mu}} \cos^{-1}\frac{p}{a}$$

Hence the required time

$$= 2t_1$$

$$= \frac{2}{\sqrt{\mu}} \cos^{-1}\frac{p}{a}$$

$$= \frac{2}{\sqrt{\mu}} \tan^{-1}\left(\frac{\sqrt{a^2-p^2}}{p}\right)$$

$$= \frac{2}{\sqrt{\mu}} \tan^{-1}\left(\frac{v}{p\sqrt{\mu}}\right)$$

$$\left[\because \text{from } ②,\ \sqrt{a^2-p^2} = \frac{v}{\sqrt{\mu}}\right]$$

$$= \frac{2}{2\pi/T} \tan^{-1}\left(\frac{v}{p(2\pi/T)}\right)$$

$$\left(\because T = \frac{2\pi}{\sqrt{\mu}} \quad \therefore \sqrt{\mu} = \frac{2\pi}{T}\right)$$

$$= \frac{T}{\pi} \tan^{-1}\left(\frac{vT}{2\pi p}\right)$$

$\therefore$ $\boxed{\text{Time elapsed before particle returns to point } P = \frac{T}{\pi} \tan^{-1}\left(\frac{vT}{2\pi p}\right)}$

To find value of P when elapsed time is $\frac{T}{2}$:

$$\therefore \frac{\cancel{T}}{2} = \frac{\cancel{T}}{\pi} \tan^{-1}\left(\frac{vT}{2\pi p}\right)$$

$$\therefore \tan\frac{\pi}{2} = \frac{vT}{2\pi p}$$

$$\infty = \frac{vT}{2\pi p}$$

$$\Rightarrow \boxed{p = 0}$$

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Q. 5 e) If $\bar{a} = \sin\theta\, i + \cos\theta\, j + \theta\, k$

$\bar{b} = \cos\theta\, i - \sin\theta\, j - 3k$

$\bar{c} = 2i + 3j - 3k$

then find the values of the derivative of the vector function $\bar{a} \times (\bar{b} \times \bar{c})$ w.r.to θ at $\theta = \frac{\pi}{2}$ and $\theta = \pi$

Sol^n $\Rightarrow$ $\bar{b} \times \bar{c} = \begin{vmatrix} i & j & k \\ \cos\theta & -\sin\theta & -3 \\ 2 & 3 & -3 \end{vmatrix}$

$= i(3\sin\theta + 9) + j(-6 + 3\cos\theta)$
$+ k(3\cos\theta + 2\sin\theta)$

$\bar{a} \times (\bar{b} \times \bar{c}) = \begin{vmatrix} i & j & k \\ \sin\theta & \cos\theta & \theta \\ 3\sin\theta + 9 & -6 + 3\cos\theta & 3\cos\theta + 2\sin\theta \end{vmatrix}$

$= i(3\cos^2\theta + 2\sin\theta\cos\theta + 6\theta -$
$3\theta\sin\theta) + j(3\theta\sin\theta + 9\theta -$
$3\sin\theta\cos\theta - 2\sin^2\theta) + k(-6\sin\theta$
$+ 3\sin\theta\cos\theta - 3\sin\theta\cos\theta - 9\cos\theta)$

$= i(3\cos^2\theta + 2\sin\theta\cos\theta - 3\theta\sin\theta$
$+ 6\theta) + j(-2\sin^2\theta - 3\sin\theta\cos\theta +$
$3\theta\sin\theta + 9\theta) + k(-6\sin\theta - 9\cos\theta)$

$\frac{d}{d\theta}(\bar{a} \times (\bar{b} \times \bar{c})) = i(-6\cos\theta\sin\theta +$
$2\cos^2\theta + - 2\sin^2\theta - 3\sin\theta - 3\theta\cos\theta + 6)$
$+ j(-2\sin\theta\cos\theta - 3\cos^2\theta + 3\sin^2\theta +$
[illegible] $2\sin\theta + 9) + k(-6\cos\theta + 9\sin\theta)$

$\theta = \frac{\pi}{2}$, $\sin\frac{\pi}{2} = 1$ $\cos\frac{\pi}{2} = 0$

$$\frac{d}{d\theta}[\bar{a}\times(\bar{b}\times\bar{c})] = i\,(-5+6) + j\,(3+3+9) + k\,(-9)$$

$$= i + 15j + 9k$$

$$\boxed{\frac{d}{d\theta}[\bar{a}\times(\bar{b}\times\bar{c})] = i + 15j + 9k}$$

$\theta = \pi$, $\sin\pi = 0$ $\cos\pi = -1$

$$\frac{d}{d\theta}[\bar{a}\times(\bar{b}\times\bar{c})] = i\,(2+3\pi+6) + j\,(-3-3\pi+9) + k\,(6)$$

$$= i\,(8+3\pi) + j\,(6-3\pi) + 6k$$

$$\boxed{\frac{d}{d\theta}[\bar{a}\times(\bar{b}\times\bar{c})] = (8+3\pi)\,i + j\,(6-3\pi) + 6k}$$

is required solution.

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Q. 6 a) Solve the differential equation:

$$\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 2y = e^x + \cos x$$

Sol^n: Given $(D^3 - 3D^2 + 4D - 2)y = e^x + \cos x$, — ①

where $D \equiv d/dx$

Its auxiliary equation is

$$D^3 - 3D^2 + 4D - 2 = 0$$

or $D^2(D-1) - 2D(D-1) + 2(D-1) = 0$

or $(D-1)(D^2 - 2D + 2) = 0$

giving $D = 1, \left(\frac{2 \pm \sqrt{4-8}}{2}\right)$

i.e. $D = 1, 1 \pm i$

$\therefore$ $\boxed{C.F. = c_1 e^x + e^x(c_2 \cos x + c_3 \sin x),}$

c_1, c_2, c_3 being arbitrary constants

P.I. corresponding to e^x

$$= \frac{1}{D^3 - 3D^2 + 4D - 2} e^x$$

$$= \frac{1}{(D-1)(D^2 - 2D + 2)} e^x$$

$$= \frac{1}{D-1} \frac{1}{1-2+2} e^x$$

$$= \frac{1}{D-1} e^x \cdot 1$$

$$= e^x \frac{1}{(D+1)-1} \cdot 1$$

$$= e^x \frac{1}{D} \cdot 1$$

$$= x e^x$$

P. I. corresponding to $\cos x$

$$= \frac{1}{D^3 - 3D^2 + 4D - 2} \cos x$$

$$= \frac{1}{D^2 \cdot D - 3D^2 + 4D - 2} \cos x$$

$$= \frac{1}{(-1^2)D - 3(-1^2) + 4D - 2} \cos x$$

$$= \frac{1}{3D+1} \cos x = (3D-1) \frac{1}{9D^2 - 1} \cos x$$

$$= (3D-1) \frac{1}{9(-1^2)-1} \cos x$$

$$= \frac{-1}{10} (3D\cos x - \cos x)$$

$$= \frac{-1}{10} (-3\sin x - \cos x) = \frac{3\sin x + \cos x}{10}$$

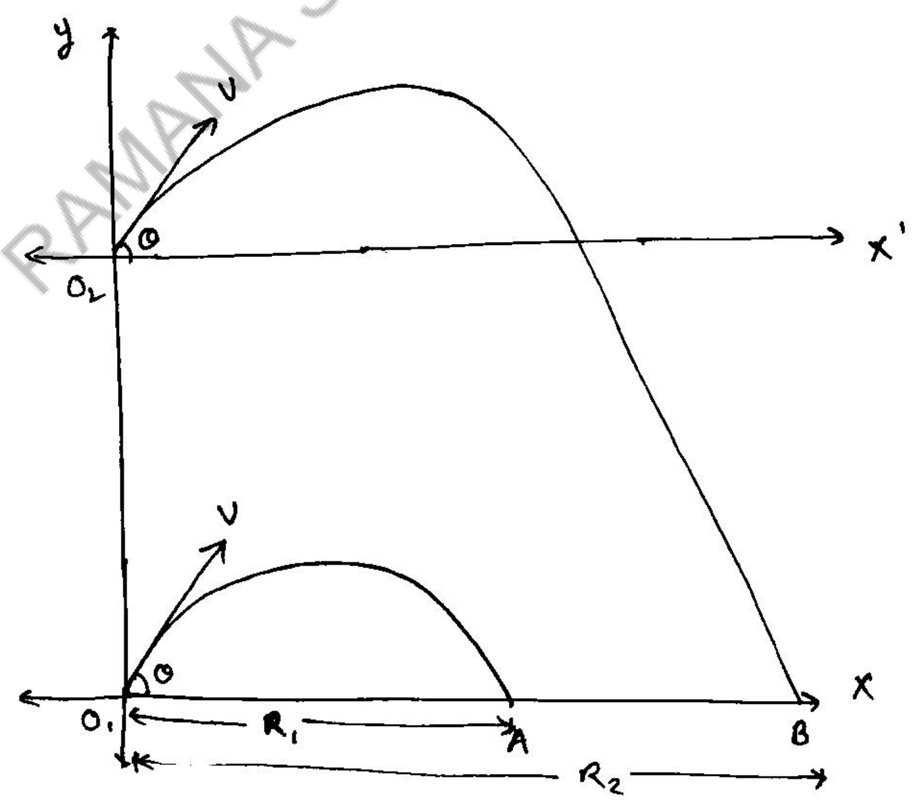
$\therefore$ Required solution is

$$\boxed{y = e^x (c_1 + c_2 \cos x + c_3 \sin x) + x e^x + \left(\frac{3\sin x + \cos x}{10} \right)}$$

Q.6 b) when a particle is projected from a point O_1 on the sea level with a velocity V and angle of Projection θ with the ~~horizontal~~ horizon in a vertical plane, its horizontal range is R_1, If it is further projected from a point O_2, which is vertically above O_1 at a height h in the same vertical plane, with the same velocity V and same angle θ with the horizon, its horizontal range is R_2. prove that $R_2 > R_1$ and $(R_2 - R_1) : R_1$ is equal to

$$\frac{1}{2}\left\{\sqrt{\left(1+\frac{2gh}{V^2\sin^2\theta}\right)}-1\right\} : 1$$

Sol^n ⇒



let R_1 be the original range. then

$$R_1 = \frac{2v^2 \sin\alpha \cos\alpha}{g}$$

let o be a point a height h above the water level. let R_2 be the range on the sea when the shot is fired from O.

Referred to the horizontal and vertical lines ox & oy in the plane of projection as the co-ordinates axes of the point Q where the shot strikes the water are ~~$(R_1, -h)$~~ (R_2, h)

the point ~~$(R_1, -h)$~~ (R_2, h) lies on the curve

$$y = x\tan\alpha - \frac{1}{2}\frac{gx^2}{v^2\cos^2\alpha}$$

$$-h = R_2\tan\alpha - \frac{1}{2}\frac{gR_2^2}{v^2\cos^2\alpha}$$

or $$R_2^2 - \frac{2}{g}v^2\sin\alpha\cos\alpha\, R_2 - \frac{2}{g}v^2 h\cos^2\alpha = 0$$

$$R_2^2 - R_2R_1 - \frac{2}{g}v^2h\cos^2\alpha = 0$$

$$R_2^2 - R_2R_1 = \frac{2}{g}v^2h\cos^2\alpha$$

$$\left(R_2 - \frac{1}{2}R_1\right)^2 = \frac{1}{4}R_2^2 + \frac{2}{g}v^2h\cos^2\alpha =$$

$$= \frac{R_1^2}{4}\left[1+\frac{1}{R_1^2}\cdot\frac{8}{g}v^2h\cos^2\alpha\right]$$

$$\therefore \left(R_2-\frac{1}{2}R_1\right)^2 = \frac{R_1^2}{4}\left[1+\frac{g^2}{4v^2\sin^2\alpha\cos^2\alpha}\cdot\frac{8}{g}v^2h\cos^2\alpha\right]$$

[by (1)]

$$= \frac{R_1^2}{4}\left[1+\frac{2gh}{v^2\sin^2\alpha}\right]$$

$$R_2-\frac{1}{2}R_1 = \frac{1}{2}R_1\left[1+\frac{2gh}{v^2\sin^2\alpha}\right]^{1/2}$$

so that,

$$R_2-R_1 = \frac{1}{2}R_1\left(1+\frac{2gh}{v^2\sin^2\alpha}\right)^{1/2}-\frac{1}{2}R$$

$$\frac{R_2-R_1}{R_1} = \frac{1}{2}\left[\left(1+\frac{2gh}{v^2\sin^2\alpha}\right)^{1/2}-1\right]$$

hence we proved required result.

Q.5 (c) Evaluate the integral $\iint_S (3y^2z^2\hat{i} + 4z^2x^2\hat{j} + z^2y^2\hat{k}) \cdot \hat{n}\, ds$ where S is the upper part of the surface $4x^2+4y^2+4z^2=1$ above the plane $z=0$ and bounded by the xy-plane. Hence verify Gauss divergence theorem.

Sol^n → by divergence theorem,

$$\iint_S (3y^2z^2 i + 4z^2x^2 j + z^2y^2 \hat{k}) \cdot \hat{n}\, ds$$

$$= \iiint_V \operatorname{div}(3y^2z^2 i + 4z^2x^2 j + z^2y^2 k)\, dV$$

where V is the volume enclosed by S.

$$= \iiint_V \left[\frac{\partial}{\partial x}(3y^2z^2) + \frac{\partial}{\partial y}(4z^2x^2) + \frac{\partial}{\partial z}(z^2y^2)\right] dv$$

$$= \iiint_V 2zy^2\, dv = 2\iiint_V zy^2\, dV$$

we shall use spherical polar co-ordinate (r, θ, ϕ) to evaluate this triple ~~trip~~ integral.

In polar,

$$dV = dr,\ r\,d\theta \cdot r\sin\theta\, d\phi$$

$$= r^2 \sin\theta\, dr\, d\theta\, d\phi.$$

~~AS1~~. ~~AS1~~ Also $z = r\cos\theta$, $y = r\sin\theta\sin\phi$

To cover V the limits of r will be 0 to $\frac{1}{2}$, those of θ will be 0 to $\frac{\pi}{2}$ and those of ϕ will be 0 to 2π.

The triple integral is

$$= 2\int_{r=0}^{1/2}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} r\cos\theta \; r^2\sin^2\theta\sin^2\phi \times r^2\sin\theta\, dr\, d\theta\, d\phi.$$

$$= 2\int_0^{1/2}\int_0^{\pi/2}\int_0^{2\pi} r^5\sin^3\theta\cos\theta\sin^2\phi\, dr\, d\theta\, d\phi$$

$$= 2\left(\frac{r^6}{6}\right)_0^{1/2}\cdot\int_0^{\pi/2}\int_0^{2\pi}\cos\theta\sin^3\theta\sin^2\phi\, d\theta\, d\phi$$

$$= 2\times\frac{1}{6}\times\frac{1}{2^6}\times\frac{2}{4.2}\int_0^{2\pi}\sin^2\phi\, d\phi$$

$$= \frac{1}{768}\times 4\times\int_0^{\pi/2}\sin^2\phi\, d\phi$$

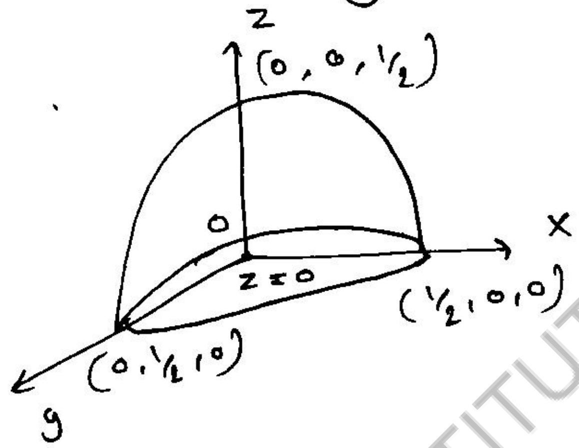
$$= \frac{1}{768}\times 4\times\frac{1}{2}\times\frac{\pi}{4}$$

$$= \frac{\pi}{768}$$

$$\boxed{V = \frac{\pi}{768}} \quad \text{———} \quad (1)$$

verification of Gauss divergence —

Given —



Given $\phi \equiv 4x^2+4y^2+4z^2-1=0$

let $\bar{F} = 3y^2z^2\, i + 4z^2x^2\, j + z^2y^2\, k$

we have, $\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = 2(xi+yj+zk)$

$F\cdot\hat{n} = 2(3xy^2z^2+4x^2z^2y+z^3y^2)$

we have $ds = \frac{dx\,dy}{|\hat{n}\cdot k|} = \frac{dx\,dy}{2z}$

$$\iint_S \bar{F}\cdot\bar{n}\, ds = \iint_R \frac{2(3x^2y^2z^2+4x^2yz^2+y^2z^3)}{2z}\, dx\,dy$$

$$= \iint_R (3xy^2z+4x^2yz+y^2z^2)\, dx\,dy \quad \text{—— (1)}$$

corresponding to $x^2+y^2 = \frac{1}{4}$

$x = r\cos\theta$; $y = r\sin\theta$

$\therefore\ dx\,dy = r\,dr\,d\theta$, $0 \le r \le \frac{1}{2}$

$0 \le \theta \le 2\pi$

$\because\ x^2+y^2+z^2=\frac{1}{4} \Rightarrow z^2=\frac{1}{4}-x^2-y^2$

$\Rightarrow z=\sqrt{\frac{1}{4}-r^2}$

$$\therefore \iint_S \bar{F}\cdot\hat{n}\,ds = \int_{r=0}^{1/2}\int_{\theta=0}^{2\pi}\Big[3r^3\cos\theta\sin^2\theta \times \sqrt{1/4-\cancel{r}^2} + 4r^3\cos^2\theta\sin\theta\sqrt{1/4-r^2} + r^2\sin^2\theta\,(1/4-r^2)\Big]\,r\,dr\,d\theta$$

$$= \int_{r=0}^{1/2}\int_{0}^{2\pi}\Big[r^4\sqrt{1/4-r^2}\,(3\cos\theta\sin^2\theta+4\cos^2\theta\sin\theta) + \frac{r^3\sin^2\theta}{4} - r^5\sin^2\theta\Big]\,dr\,d\theta$$

$$= \int_0^{2\pi}\Big[\sin^2\theta\Big(\frac{1}{16\times 2^4}-\frac{1}{2^{\cancel{2}6}\times 6}\Big)\Big]\,d\theta + \int_{r=0}^{1/2} r^4\sqrt{1/4-r^2}\Big(\int_{\theta=0}^{2\pi} 3\cos\theta\sin^2\theta + 4\cos^2\theta.\sin\theta\Big)\,d\theta$$

$$= \frac{\pi}{768} \qquad \text{------------ (2)}$$

by (1) & (2) we verified Gauss divergence theorem.

Q. 7 a) i) Find the solution of the differential equation: $\frac{dy}{dx} = -\frac{2xy^3+2}{3x^2y^2+8e^{4y}}$.

Sol^n:

$$\frac{dy}{dx} = -\frac{2xy^3+2}{3x^2y^2+8e^{4y}}$$

$$(2xy^3+2)\,dx + (3x^2y^2+8e^{4y})\,dy = 0$$

$$M\,dx + N\,dy = 0$$

$$\therefore M = 2xy^3+2 \qquad N = 3x^2y^2+8e^{4y}$$

$$\frac{\partial M}{\partial y} = 6xy^2 \; ; \; \frac{\partial N}{\partial x} = 6xy^2$$

$$\therefore \frac{\partial M}{\partial y} = 6xy^2 = \frac{\partial N}{\partial x}$$

Hence above ODE is exact.

Therefore solution is

$$\int_{y=c} M\,dx + \int_{\text{no } x \text{ term}} N\,dy = c$$

$$\int_{y=c} (2xy^3+2)\,dx + \int 8e^{4y}\,dy = c$$

$$\Rightarrow 2y^3\frac{x^2}{2} + 2x + 8\frac{e^{4y}}{4} = c$$

$$\boxed{x^2y^3 + 2x + 2e^{4y} = c}$$

which is the required solution.

Q. 7 a) ii) Reduce the equation $x^2p^2 + y(2x+y)p + y^2 = 0$ to Clairaut's form by the substitution $y = u$ and $xy = v$. Hence solve the equation and show that $y + 4x = 0$ is a singular solution of the differential equation.

Sol^n: Given equation is $x^2p^2 + yp(2x+y) + y^2 = 0$ - ①

Given $y = u$ and $xy = v$. — ②

Differentiating ②

$$dy = du \quad \text{and} \quad x\,dy + y\,dx = dv$$

$$\therefore \frac{x\,dy + y\,dx}{dy} = \frac{dv}{du}$$

or $x + y\dfrac{dx}{dy} = \dfrac{dv}{du}$

or $x + \dfrac{y}{p} = P$

or $\dfrac{y}{p} = P - x$

or $p = \dfrac{y}{(P-x)}$ where $p = \dfrac{dy}{dx}$, $P = \dfrac{dv}{du}$

putting $p = y/(P-x)$ in ①, we have

$$\frac{x^2y^2}{(P-x)^2} + \frac{y^2}{P-x}(2x+y) + y^2 = 0$$

or $x^2 + (P-x)(2x+y) + (P-x)^2 = 0$

or $\quad py - xy + p^2 = 0$

or $\quad \boxed{v = up + p^2}$ using ② $\quad$ — ③

③ is in Clairaut's form.

So replacing p by c its general solution is

$$v = uc + c^2$$

or $\quad xy = yc + c^2$, $\quad c$ being an arbitrary constant.

or $\quad c^2 + cy - xy = 0$

which is a quadratic equation in c

and hence c-discriminant relation is

$$y^2 - 4 \cdot 1 \cdot (-xy) = 0$$

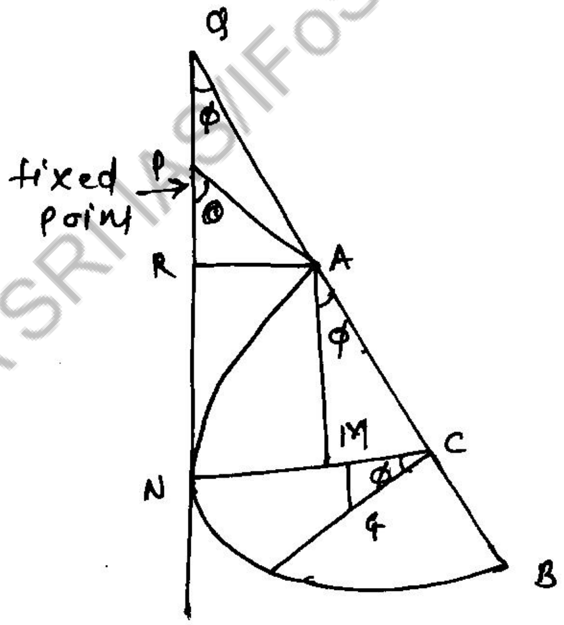
or $\quad y(y + 4x) = 0$

Since $y = 0$ and $\underline{y + 4x = 0}$ both satisfy ①,

so these are both $\underline{\text{singular solutions}}$.

Q.7 (b) A solid hemisphere is supported by a string fixed to a point on its rim and to a point on a smooth vertical wall with which the curved surface is in contact. If θ is the angle of inclination of the string with vertical and ϕ is the angle of inclination of the plane base of the hemisphere to the vertical, then find the value of $(\tan\phi - \tan\theta)$.

sol^n ⇒



let the length of the string $AP = l$

and let the radius of the hemisphere be a so that $CG = \frac{3a}{8}$

P is a fixed point in the wall.

Give a small displacement to the system such that θ and ϕ become $\theta + \delta\theta$ and $\phi + \delta\phi$ respectively.

Equation of virtual work is

$w\delta$ (deepth of G below P) = 0

or δ (depth of G below P) = 0

$$\delta(PR + RN + MG) = 0$$

or $\delta\left(l\cos\theta + a\cos\phi + \frac{3a}{8}\sin\phi\right) = 0$

or $-l\sin\theta\,\delta\theta + \delta\phi\left(-a\sin\phi + \frac{3a}{8}\cos\phi\right) = 0$ .... (1)

from the figure it is clear that

$CN = CL + LN = a\sin\phi + l\sin\theta$

$a = a\sin\phi + l\sin\theta$

taking differential $a\cos\phi\cdot\delta\phi + l\cos\theta\,\delta\theta = 0$

$$\frac{\delta\theta}{\delta\phi} = -\frac{a}{l}\frac{\cos\phi}{\cos\theta}$$

using this in (1) we get

$$-l\sin\theta\cdot\left(-\frac{a\cos\phi}{l\cos\theta}\right) + \left(-a\sin\phi + \frac{3a}{8}\cos\theta\right) = 0$$

$$\tan\theta \cdot \cos\phi - \sin\phi + \frac{3}{8}\cos\phi = 0$$

dividing by $\cos\phi$,

$$\tan\theta - \tan\phi + \frac{3}{8} = 0$$

$$\boxed{\tan\phi - \tan\theta = \frac{3}{8}}$$

Hence required solution.

Q. 7 c) If the tangent to a curve makes a constant angle θ with a fixed line, then prove that the ratio of radius of torsion to radius of curvature is proportional to $\tan\theta$. Further prove that if this ratio is constant, then the tangent makes a constant angle with a fixed direction.

Sol^n: Let's consider a curve in 3-D space parametrized by arc length, denoted by $\vec{r}(s)$, where s is the arc length parameter. The radius of curvature R is given by:

$$R = \left|\frac{d\vec{T}}{ds}\right|^{-1}$$

where $\vec{T}$ is the unit tangent vector to the curve.

Now, let's consider the tangent vector $\vec{T}(s)$ at a point on the curve. According to the problem, the tangent makes a constant angle θ with a fixed line. This means that $\vec{T}(s)$ is rotating about a fixed axis with a constant angular velocity ω, such that

$$\theta = \omega s$$

Differentiating $\vec{T}(s)$ with respect to s gives us $\frac{d\vec{T}}{ds} = \omega \vec{N}$, where $\vec{N}$ is the unit normal vector.

Now, the radius of torsion T is defined as the reciprocal of the rate of change of θ w.r.t. arc length s:

$$T = \left|\frac{d\theta}{ds}\right|^{-1} = \left|\frac{d(\omega s)}{ds}\right|^{-1} = |\omega|^{-1}$$

Now, the radiu

So, we have found expressions for both the radius of curvature and the radius of torsion.

Now, let's examine the relationship between T and R:

$$\frac{T}{R} = |\omega|^{-1} \cdot \left|\frac{d\vec{T}}{ds}\right|^{-1}$$

Since $\left|\frac{d\vec{T}}{ds}\right| = |\omega \vec{N}| = |\omega| \cdot |\vec{N}| = |\omega|$,

we can simplify the expression:

$$\boxed{\frac{T}{R} = |\omega|^{-1} \cdot |\omega|^{-1} = |\omega|^{-2} = \tan^2(\theta)}$$

So, we have proved that the ratio of the radius of torsion to radius of curvature is proportional to $\tan^2(\theta)$.

Now, let's prove the second part of the statement.

If this ratio $\frac{I}{R}$ is constant, it implies that $\tan^2(\theta)$ is constant. Since $\tan^2(\theta)$ is constant, θ must be constant as well, because the tangent of a constant angle is also a constant. Therefore, if the ratio $\frac{I}{R}$ is constant, the tangent makes a constant angle θ with a fixed direction.

Q. 8 a) Solve the following initial value problem by using Laplace transform technique:

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 3y(t) = f(t),$$

$y(0)=1$, $y'(0)=0$ and $f(t)$ is a given function of t.

Sol^n: Given $y'' - 4y' + 3y = f(t)$ — ①

where $y'' = \frac{d^2y}{dt^2}$, $y' = \frac{dy}{dt}$

with initial condition

$y(0) = 1$ and $y'(0) = 0$ — ②

Taking Laplace transform of both sides of ①, we get

$$L\{y''\} - 4L\{y'\} + 3L\{y\} = L\{f(t)\}$$

or $s^2 L\{y\} - s\,y(0) - y'(0) - 4[sL\{y\} - y(0)]$
$+ 3L(y) = f(s)$ — ③

where $L\{f(t)\} = f(s)$ so that $L^{-1}\{f(s)\} = f(t)$ — ④

Using ②,

③ $\Rightarrow (s^2 - 4s + 3)L\{y\} - s + 4 = f(s)$

or $L\{y\} = \dfrac{s - 4 + f(s)}{(s^2 - 4s + 3)}$

$$= \frac{s-4}{(s-1)(s-3)} + \frac{f(s)}{(s-1)(s-3)}$$

or $L\{y\} = \frac{1}{2}\left[\frac{3}{s-1} - \frac{1}{s-3}\right] + \frac{1}{2} f(s)\left[\frac{1}{s-3} - \frac{1}{s-1}\right]$

on resolving into partial fractions

$\therefore y = \frac{3}{2} L^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{2} L^{-1}\left\{\frac{1}{s-3}\right\} + \frac{1}{2} L^{-1}\left\{f(s)\frac{1}{s-3}\right\}$
$\quad - \frac{1}{2} L^{-1}\left\{f(s)\frac{1}{s-1}\right\}$

$y = \frac{3}{2} e^{t} - \frac{1}{2} e^{3t} + \frac{1}{2} L^{-1}\{f(s)g(s)\}$
$\quad - \frac{1}{2} L^{-1}\{f(s)h(s)\}$ — (5)

where $g(s) = \frac{1}{s-3}$ and $h(s) = \frac{1}{s-1}$ — (6)

so that $g(t) = L^{-1}\{g(s)\} = e^{3t}$
and $h(t) = L^{-1}\{h(s)\} = e^{t}$ } (7)

Now, by using the convulation theorem and (7), we have

$L^{-1}\{f(s)g(s)\} = \int_0^t f(u)g(t-u)\,du$

$= \int_0^t f(u)\, e^{3(t-u)}\,du$

$= e^{3t}\int_0^t f(u)\, e^{-3u}\,du$ — (8)

and $L^{-1}\{f(s)h(s)\} = \int_0^t f(u)\, h(t-u)\,du$

$= \int_0^t f(u)\, e^{t-u}\,du$

$= e^{t}\int_0^t f(u)\, e^{-u}\,du$ — (9)

Using (8) and (9), (5) reduces to

$$\boxed{y = \frac{1}{2}(3e^{t} - e^{3t}) + \frac{1}{2}e^{3t}\int_0^t f(u)e^{-3u}\,du - \frac{1}{2}e^{t}\int_0^t f(u)e^{-u}\,du}$$

Q. 8. b) A particle is projected from an apse at a distance $\sqrt{c}$ from the centre of force with a velocity $\sqrt{\frac{2\lambda c^3}{3}}$ and is moving with central acceleration $\lambda(r^5 - c^2 r)$. Find the path of motion of this particle. Will that be the curve $x^4 + y^4 = c^2$?

Sol^n: Here the central acceleration

$$P = \lambda(r^5 - c^2 r) = \lambda\left(\frac{1}{u^5} - \frac{c^2}{u}\right)$$

$\therefore$ The differential equation of the path is

$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2}$$

$$= \frac{\lambda}{u^2}\left(\frac{1}{u^5} - \frac{c^2}{u}\right)$$

$$= \lambda\left(\frac{1}{u^7} - \frac{c^2}{u^3}\right)$$

Multiplying both sides by $2\left(\frac{du}{d\theta}\right)$ and then integrating, we have

$$v^2 = h^2\left(u^2 + \left(\frac{du}{d\theta}\right)^2\right) = \lambda\left[\frac{-1}{3u^6} + \frac{c^2}{u^2}\right] + A \quad -(1)$$

where A is constant.

But initially, when $r = \sqrt{c}$ i.e. $u = \frac{1}{\sqrt{c}}$

$\frac{du}{d\theta} = 0$ (at an apse) and

$$v = \sqrt{\frac{2\lambda}{3} c^3}$$

$\therefore$ from (1), we have

$$\left(\sqrt{\frac{2\lambda c^3}{3}}\right)^2 = h^2\left(\frac{1}{\sqrt{c}}\right)^2 = \lambda\left(-\frac{c^3}{3}+c^3\right)+A$$

$$\frac{2\lambda c^3}{3} = \frac{h^2}{c} = \lambda\frac{2c^3}{3} + A$$

$$\therefore\ A = 0 \text{ and } h^2 = \frac{2\lambda c^4}{3}$$

Substituting the values of h^2 and A in ①, we have

$$\frac{2\lambda}{3}c^4\left[u^2+\left(\frac{du}{d\theta}\right)^2\right] = \lambda\left(\frac{-1}{3u^6}+\frac{c^2}{u^2}\right)$$

$$c^4\left[u^2+\left(\frac{du}{d\theta}\right)^2\right] = \frac{3}{2}\left(\frac{-1}{3u^6}+\frac{c^2}{u^2}\right)$$

$$c^4\left(\frac{du}{d\theta}\right)^2 = \frac{-1}{2u^6}+\frac{3c^2}{2u^2} - c^4u^2$$

$$= \frac{1}{u^6}\left[\frac{-1}{2}+\frac{3}{2}c^2u^4 - c^4u^8\right]$$

$$= \frac{1}{u^6}\left[\frac{-1}{2} - \left(c^4u^8 - \frac{3}{2}c^2u^4\right)\right]$$

$$= \frac{1}{u^6}\left[\frac{-1}{2} - \left(c^2u^4-\frac{3}{4}\right)^2 + \frac{9}{16}\right]$$

$$= \frac{1}{u^6}\left[\frac{1}{16} - \left(c^2u^4-\frac{3}{4}\right)^2\right]$$

$$= \frac{1}{u^6}\left[\left(\frac{1}{4}\right)^2 - \left(c^2u^4-\frac{3}{4}\right)^2\right]$$

$$\therefore\ c^2u^3\left(\frac{du}{d\theta}\right) = \sqrt{\left(\frac{1}{4}\right)^2 - \left(c^2u^4-\frac{3}{4}\right)^2}$$

$$d\theta = \frac{c^2u^3\,du}{\sqrt{\left(\frac{1}{4}\right)^2-\left(c^2u^4-\frac{3}{4}\right)^2}} \quad -(2)$$

Putting $c^2u^4-\frac{3}{4}=z \Rightarrow 4c^2u^3\,du = dz$

$c^2u^3\,du = \frac{1}{4}dz$

$\therefore (2) \Rightarrow 4\,d\theta = \dfrac{dz}{\sqrt{\left(\frac{1}{4}\right)^2-z^2}}$

Integrating,

$4\theta + B = \sin^{-1}\left(\dfrac{z}{1/4}\right) = \sin^{-1}(4z)$

where B is constant.

$4\theta + B = \sin^{-1}(4c^2u^4-3)$

But initially when $u=\frac{1}{\sqrt{c}}$, $\theta=0$

$\therefore B = \sin^{-1}(1) = \frac{\pi}{2}$

$\therefore 4\theta+\frac{\pi}{2} = \sin^{-1}(4c^2u^4-3)$

$\sin\left(4\theta+\frac{\pi}{2}\right) = 4c^2u^4-3$

$\cos 4\theta = 4c^2u^4-3$

$4c^2u^4 = 3+\cancel{4\cos\theta}\ \cos 4\theta$

$4\dfrac{c^2}{r^4} = 3+\cos 4\theta$

$4c^2 = r^4[3+(2\cos^2 2\theta-1)]$

$4c^2 = 2r^4[1+\cos^2 2\theta]$

$4c^2 = 2r^4[(\cos^2\theta+\sin^2\theta)^2+(\cos^2\theta-\sin^2\theta)^2]$

$$4c^2 = 4r^4\left[\cos^4\theta + \sin^4\theta\right]$$

$$c^2 = (r\cos\theta)^4 + (r\sin\theta)^4$$

$$\boxed{c^2 = x^4 + y^4}$$

which is the required equation of the path.

Q. 8. c) For a scalar point function ϕ and vector point function $\vec{f}$, prove the identity $\nabla\cdot(\phi\vec{f}) = \nabla\phi\cdot\vec{f} + \phi(\nabla\cdot\vec{f})$. Also find the value of $\nabla\cdot\left(\frac{f(r)}{r}\vec{r}\right)$ and then verify stated identity.

Sol^n: i) We have,

$$\text{div}(\phi\vec{f}) = \nabla\cdot(\phi\vec{f})$$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot(\phi\vec{f})$$

$$= i\cdot\frac{\partial}{\partial x}(\phi\vec{f}) + j\cdot\frac{\partial}{\partial y}(\phi\vec{f}) + k\cdot\frac{\partial}{\partial z}(\phi\vec{f})$$

$$= \Sigma\left\{i\cdot\left(\frac{\partial}{\partial x}(\phi\vec{f})\right)\right\}$$

$$= \Sigma\left\{i\cdot\left(\frac{\partial\phi}{\partial x}\vec{f} + \phi\frac{\partial\vec{f}}{\partial x}\right)\right\}$$

$$= \Sigma\left\{i\cdot\left(\frac{\partial\phi}{\partial x}\vec{f}\right)\right\} + \Sigma\left\{i\cdot\left(\phi\frac{\partial\vec{f}}{\partial x}\right)\right\}$$

$$= \Sigma\left\{\left(\frac{\partial\phi}{\partial x}i\right)\cdot\vec{f}\right\} + \Sigma\left\{\phi\left(i\cdot\frac{\partial\vec{f}}{\partial x}\right)\right\}$$

$$\left[\because a\cdot(mb) = (ma)\cdot b = m(a\cdot b)\right]$$

$$= \left\{\Sigma\frac{\partial\phi}{\partial x}i\right\}\cdot\vec{f} + \phi\,\Sigma\left(i\cdot\frac{\partial\vec{f}}{\partial x}\right)$$

$$= \underline{\nabla\phi\cdot\vec{f} + \phi(\nabla\cdot\vec{f})}$$

Hence proved

ii) $\nabla \cdot \left(\frac{f(r)}{r} \vec{r} \right)$

$$= \nabla \cdot \left\{ \frac{f(r)}{r} (xi + yj + zk) \right\}$$

$$= \frac{\partial}{\partial x}\left\{ \frac{f(r)}{r} x \right\} + \frac{\partial}{\partial y}\left\{ \frac{f(r)}{r} y \right\} + \frac{\partial}{\partial z}\left\{ \frac{f(r)}{r} z \right\} \quad -(1)$$

Now $\frac{\partial}{\partial x}\left\{ \frac{f(r)}{r} x \right\} = \frac{f(r)}{r} + x \frac{d}{dr}\left\{ \frac{f(r)}{r} \right\} \frac{\partial r}{\partial x}$

$$= \frac{f(r)}{r} + x \left\{ \frac{f'(r)}{r} - \frac{1}{r^2} f(r) \right\} \frac{x}{r}$$

$$= \frac{f(r)}{r} + \frac{x^2}{r^2} f'(r) - \frac{x^2}{r^3} f(r)$$

Similarly $\frac{\partial}{\partial y}\left\{ \frac{f(r)}{r} y \right\} = \frac{f(r)}{r} + \frac{y^2}{r^2} f'(r) - \frac{y^2}{r^3} f(r)$

and $\frac{\partial}{\partial z}\left\{ \frac{f(r)}{r} z \right\} = \frac{f(r)}{r} + \frac{z^2}{r^2} f'(r) - \frac{z^2}{r^3} f(r)$

putting these values in (1), we get,

$$\nabla \cdot \left(\frac{f(r)}{r} \vec{r} \right) = \frac{3}{r} f(r) + \frac{r^2}{r^2} f'(r) - \frac{r^2}{r^3} f(r)$$

$$= \frac{2}{r} f(r) + f'(r)$$

$$= \frac{1}{r^2} \left[2r f(r) + r^2 f'(r) \right]$$

$$= \frac{1}{r^2} \frac{d}{dr} \left[r^2 f(r) \right] \quad -(2)$$

To verify stated :-

$$\phi = \frac{f(r)}{r} \qquad \vec{f} = \vec{r}$$

$$\nabla\cdot(\phi\vec{f}) = \nabla\cdot\left(\frac{f(r)}{r}\vec{r}\right)$$

$$= \frac{1}{r^2}\frac{d}{dr}[r^2 f(r)] \quad \text{- from } 2$$

i.e. $L.H.S = \nabla\cdot(\phi\vec{f}) = \frac{1}{r^2}\frac{d}{dr}[r^2 f(r)]$

$$R.H.S. = \nabla\phi\cdot\vec{f} + \phi(\nabla\cdot\vec{f})$$

$$= \nabla\left(\frac{f(r)}{r}\right)\cdot\vec{r} + \frac{f(r)}{r}(\nabla\cdot\vec{r})$$

$$= \left[\nabla f(r)\left(\frac{1}{r}\right) + f(r)\,\nabla\left(\frac{1}{r}\right)\right]\cdot\vec{r} + 3\frac{f(r)}{r}$$

$$= \left[\frac{f'(r)}{r}\nabla r + f(r)\left(\frac{-\vec{r}}{r^3}\right)\right]\cdot\vec{r} + 3\frac{f(r)}{r}$$

$$= \left[\frac{f'(r)}{r^2}\vec{r} - \frac{f(r)}{r^3}\vec{r}\right]\cdot\vec{r} + 3\frac{f(r)}{r}$$

$$= \left[\frac{f'(r)}{r^2} - \frac{f(r)}{r^3}\right](\vec{r}\cdot\vec{r}) + 3\frac{f(r)}{r}$$

$$= \left(\frac{f'(r)}{r^2} - \frac{f(r)}{r^3}\right)r^2 + 3\frac{f(r)}{r}$$

$$= f'(r) - \frac{f(r)}{r} + 3\frac{f(r)}{r}$$

$$= f'(r) + \frac{2}{r}f(r)$$

$$= \frac{1}{r^2}\left[r^2 f'(r) + 2r\, f(r)\right]$$

$$= \frac{1}{r^2}\frac{d}{dr}\left[r^2 f(r)\right]$$

$\therefore$ L.H.S. = R.H.S.

Hence stated identity

i.e. $\nabla\cdot(\phi\vec{f}) = \nabla\phi\cdot\vec{f} + \phi(\nabla\cdot\vec{f})$

is verified.