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A Note on the Principle of Transference

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Abstract

We give a precise statement of the Principle of Transference. The proof is a simple consequence of a Clifford algebra construction due to Study [7], and explained in Porteous [5]. It applies in all odd dimensions. The construction gives a realization of the double cover of the group of inhomogeneous rotations as a sub-group of the group of units of a certain Clifford algebra. This Clifford algebra also contains the Lie algebra of the inhomogeneous rotation group. In three dimensions a study of the invariant bilinear forms on the Lie algebra leads us to a co-ordinate free description of the pitch of a screw.

1. Introduction

The principle of transference has the status of a "folk" theorem. It was reputedly proved by Kotelnikov, but this reference is said to have been destroyed during the Russian revolution. Consequently there is no precise statement of the theorem and so no understanding of its range of validity.

It has long been known that lines in 3-space can be represented by dual vectors. That is to say elements of $\mathbb{D}^3$, where $\mathbb{D}$ is the ring of dual numbers $\mathbb{D} = \mathbb{R}[\varepsilon]/\{\varepsilon^2 = 0\}$. Now roughly speaking the principle of transference says that expressions concerning systems of position vectors remain true for systems of lines if the vectors are replaced by the corresponding dual vectors, see [6]. The utility of the principle for the theory of spatial mechanisms is clear. However for a deeper understanding of the principle we must set it in a more general context. So for our purposes we will adopt the following definition.

Principle of Transference

> If M is a representation module for the rotation group $SO(3)$ then $M \otimes \mathbb{D}$ is a representation module for the group of proper euclidean motions (inhomogeneous rotations) $SO(3) \circledS \mathbb{R}^3$.

We could have been even more general here since as we shall see, the principle applies to all odd dimensions. The relevance of representations of $SO(3)$ and $SO(3) \circledS \mathbb{R}^3$ is clear if we assume that "expressions concerning systems of vectors or lines" behave covariantly under the relevant symmetry operation. This is almost guaranteed to be the case since physically relevant quantities should not depend on the co-ordinate system from which they are derived.

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In order to prove the principle of transference, we first study the Clifford algebra of a certain degenerate quadratic form. This is well worth the effort, not only does it reduce the proof of the principle of transference to a simple observation but also we get a representation of the Lie algebra of the inhomogeneous group. The Lie algebra of a group can be thought of as its infinitesimal elements. In three dimensions we can identify elements of the Lie algebra with motors [1], and the projective motors are screws.

This view of motors and screws seems to be novel. We hope to explore its implications more deeply in future papers.

2. The Clifford Algebra

We follow closely the treatment given by Porteous [5 p.276]. Let $\mathbb{R}_{p,q}$ denote the usual Clifford algebra with p basis elements which square to +1 and q basis elements which square to -1. We prolong $\mathbb{R}_{p,q}$ by adding an extra basis element e, which anti-commutes with all the other basis elements but squares to zero; $e^2 = 0$. We label the resulting algebra $\mathbb{R}_{p,q,1}$. Now in analogy with the Spin groups we look at a sub-group of the units in $\mathbb{R}_{0,n,1}$ and examine its action on the $\mathbb{R}^n$ of monomials in $e_1, \ldots, e_n$.

Consider the group $E(n) :=$
$\{(g + \tfrac{1}{2}tge) \in \mathbb{R}_{0,n,1} : g \in \mathrm{Spin}(n);\ t \in \mathbb{R}^n\}$
this is clearly a sub-group of the even sub algebra, $\mathbb{R}^+_{0,n,1}$ of $\mathbb{R}_{0,n,1}$. For any $x \in \mathbb{R}^n$ we have

$$(g + \tfrac{1}{2}tge)\ (1 + xe)\ (g^- + \tfrac{1}{2}g^- te)$$
$$= 1 + f(x)e \qquad (1)$$

where $^-$ is the conjugation in $\mathbb{R}_{0,n}$ and f is a rigid motion of $\mathbb{R}^n$. The map f is readily calculated

$$f(x) = gxg^- + t$$

The product of two group elements is

$$(g + \tfrac{1}{2}tge)\ (g' + \tfrac{1}{2}t'g'e)$$
$$= (gg' + \tfrac{1}{2}(t + gt'g^-)gg'e) \qquad (2)$$

Clearly E(n) is the semi-direct product;

$$E(n) = \mathrm{Spin}(n) \circledS \mathbb{R}^n$$

Note that if we chose $g \in \mathrm{Pin}(n)$ the corresponding group defined as above would be the semi-direct product of Pin(n) with $\mathbb{R}^n$. However because of (2) this would not give the usual action of Pin(n) on $\mathbb{R}^n$. Also we can regard (1) as giving a homomorphism of E(n) onto $SO(n) \circledS \mathbb{R}^n$. The only elements of E(n) which give the identity rigid motion are 1 and -1.

Now we observe that for even dimensions we have the following isomorphism

$$\mathbb{R}_{0,2k,1} = \mathbb{R}_{0,2k} \otimes \mathbb{D}$$

Writing ε for $1 \otimes \varepsilon$ and e_i for $e_i \otimes 1$ the isomorphism is given by sending $e_i \longmapsto e_i$ and $e_1 \ldots e_{2k}e \longmapsto \varepsilon$. This is an algebra isomorphism so the image of any element is known once the images of the generators are given; for example we have:

$$e_i e_j \longmapsto e_i e_j$$
and
$$e_2 \ldots e_{2k}e = (-e_1)(e_1 \ldots e_{2k}e) \longmapsto -e_1\varepsilon$$

This only works for even dimensions since only then does each e_i commute with $e_1 \ldots e_{2k}e$.

Now we are in a position to prove the principle of transference. Any SO(2k + 1)-module M carries a representation of Spin(2k + 1) and so a representation of the algebra $\mathbb{R}^+_{0,2k+1}$. The even subalgebra of a Clifford algebra is isomorphic to the Clifford algebra with one less generator, see [5 p.253]

$$\mathbb{R}^+_{0,2k+1} = \mathbb{R}_{0,2k}$$

Clearly $M \otimes \mathbb{D}$ carries a representation of $\mathbb{R}_{0,2k} \otimes \mathbb{D} = \mathbb{R}_{0,2k,1}$ and so by restriction is a module for the subgroup $\mathrm{Spin}(2k+1) \circledS \mathbb{R}^{2k+1}$. Again by restriction this must also be a module for $SO(2k+1) \circledS \mathbb{R}^{2k+1}$ since M was originally a $SO(2k+1)$-module.

We conclude this section with a few remarks. For $k = 1$ (i.e. $SO(3)$) the relevant algebra is $\mathbb{R}_{0,2} \otimes \mathbb{D} = \mathbb{H} \otimes \mathbb{D}$ which is just the dual quaternions, also sometimes called Clifford's biquaternions.

Our proof uses the spin groups and so it is clear that the Principle of Transference also applies to the double valued or spinor representations. In particular the quaternion representation of $SO(3)$ ($\mathrm{Spin}(3) \cong SU(2)$) becomes the dual quaternion representation of the inhomogeneous group. In the above we would have got exactly the same results if we had started with $\mathbb{R}_{n,o}$ instead of $\mathbb{R}_{o,n}$, our choice reflects the conventions of the subject, see [2] for example. Finally we note that by no means all of real representations of the inhomogeneous group are obtained by this dualizing construction. A thorough treatment of the representation theory of $SO(3) \circledS \mathbb{R}^3$ can be found in [4].

For examples of the use of the Principle of Transference in kinematics see [3]. Here we will content ourselves with a few remarks that expand on the definition given in the last section. Consider two vectors $\underline{A} = (A_1, A_2, A_3)^T$ and $\underline{B} = (B_1, B_2, B_3)^T$, by definition these transform according to the standard 3-dimensional representation of $SO(3)$. Now the dot product $\underline{A} \cdot \underline{B}$ is an invariant, that is it transforms according to the trivial 1-dimensional representation. The cross product $\underline{A} \wedge \underline{B}$, transforms as a vector, this is special to $SO(3)$ and it also has other special properties; see the next section. However some aspects of the cross product can be generalised to $2k+1$ dimensions, we can form the second order antisymmetric tensor $T_{ij} = \frac{1}{2}(A_i B_j - A_j B_i)$. This transforms according to a linear representation of $SO(2k+1)$ and hence we are justified in calling it a tensor. Now we can also form a symmetric second order tensor, $S_{ij} = \frac{1}{2}(A_i B_j + A_j B_i)$, sometimes called a symmetric dyad. All these objects and their generalisations are co-ordinate free, in the sense that if we know their values in one co-ordinate system we can use their transformation properties to find their values in a transformed co-ordinate system. We expect physically meaningful quantities to behave in this way. Moreover we expect to be able to express physically meaningful laws and formulae in terms of such objects. Now the Principle of Transference as we have proved it above, tells us that for each rotationally invariant, covariant or contravariant quantity there is a corresponding dualized quantity which transforms according to the dualized representation. The dualized quantity is hence physically meaningful for systems which are symmetric with respect to the inhomogenous group. Dual vectors correspond to vectors and dual dot and cross products are the dualizations of the ordinary dot and cross product as is well known. We have shown that this is generally true, for example dual tensors can be defined and would transform according to the dualization of the ordinary tensor representations.

3. The Lie Algebra of the Inhomogeneous Rotation Group

We conclude this note with a brief account of the Lie algebra of $SO(n) \circledS \mathbb{R}^n$. This will give an indication of the power of the Clifford algebra construction.

The Lie algebra $\mathcal{L}\mathrm{Spin}(n)$, of the group $\mathrm{Spin}(n)$ is generated as a vector space, by the monomials of order 2 in $\mathbb{R}_{0,n}$. The same is true for the Lie algebra $\mathcal{L}E(n)$ of $E(n)$, that is a basis is given by the elements:

$$L_{ij} = \tfrac{1}{2} e_i e_j \qquad N_i = \tfrac{1}{2} e_i e; \qquad i \neq j$$

The product is just the commutator with respect to the Clifford multiplication

$$[X,Y] = XY - YX \quad ; \quad X,Y \in \mathcal{L}E(n)$$

The adjoint action of the group on its Lie algebra is given as follows

$$Ad_{\gamma}(X) = \gamma X \gamma^{-1} \qquad \gamma \in E(n), \quad X \in \mathcal{L}E(n)$$

Elements of Spin(n) simply act as rotations

$$Ad_g(L_{ij}) = \tfrac{1}{2} g e_i e_j g^{-}, \quad Ad_g(N_i) = \tfrac{1}{2} g e_i g^{-} e \qquad g \in Spin(n)$$

Translations have the following effect

$$Ad_t(L_{ij}) = (1 + \tfrac{1}{2}te)\, \tfrac{1}{2} e_i e_j\, (1 - \tfrac{1}{2}te)$$

$$= \tfrac{1}{2} e_i e_j + \tfrac{1}{2}(t e_i e_j - e_i e_j t)e$$

$$Ad_t(N_i) = (1 + \tfrac{1}{2}te)\, \tfrac{1}{2} e_i e\, (1 - \tfrac{1}{2}te)$$

$$= \tfrac{1}{2} e_i e$$

For $n = 3$ this confirms our identification of motors with elements of the Lie algebra. In this dimension we have the isomorphism

$$h : \mathcal{L}E(3) \longrightarrow \mathbb{R}^3 \otimes \mathbb{D}$$

$h(L_{12}) = V_3$; + cyclic ; $h(N_i) = V_i$ where $\{V_1, V_2, V_3\}$ generates $\mathbb{R}^3$. The product is given by the dual vector product

$$h([X,Y]) = h(X) \wedge h(Y)$$

So the term $\tfrac{1}{2}(tee_ie_j - e_ie_jte)$ above corresponds to the moment of the translation, see [1].

Next consider the invariant bilinear symmetric forms on the Lie algebra. In general we have a Lie algebra $\mathcal{L}\Gamma$ of a Lie group Γ, with $\{x_i\}$ a basis for $\mathcal{L}\Gamma$. Bilinear forms q, can be identified with matrices

$$q(x_i, x_j) = \sum_{ij} x_i Q_{ij} x_j$$

and symmetric forms are matrices which satisfy $Q_{ij} = Q_{ji}$. The set of all these matrices has the structure of a vector space with dimension $m = \tfrac{1}{2}(d+1)d$ where $d = \dim \mathcal{L}\Gamma$, call it $Mat^s(m)$.

The adjoint action of Γ on $\mathcal{L}\Gamma$ is linear and so can also be written as a matrix

$$Ad_{\gamma}(x_j) = (Ad_{\gamma})_{ij} x_j \qquad \gamma \in \Gamma, \; x_j \in \mathcal{L}\Gamma$$

This induces the following action on $Mat^s(m)$ the symmetric matrices

$$\gamma : Q_{ij} \mapsto \sum_{ij} (Ad_{\gamma})_{ki} Q_{ij} (Ad_{\gamma})_{je}$$

$$, Q_{ij} \in Mat^s(m), \; \gamma \in \Gamma$$

The action is linear so we have a representation of Γ on $Mat^s(m)$. The sub-vector space on which this representation acts trivially corresponds to the invariant bilinear symmetric forms. For SO(n) it is well known [8] that the only invariant form is the dot product. This is also, of course, the Killing form.

Now when n=3 we have seen that we have

$$\mathcal{L}E(3) = \mathcal{L}SO(3) \otimes \mathbb{D}$$

So that the adjoint representation of E(3) is just the "dualization" of the adjoint representation of SO(3), that is the representation given by the principle of transference. Similarly it is clear that the representation of E(3) on $Mat^s(6) \otimes \mathbb{D}$ is induced from the representation of SO(3) on $Mat^s(6)$. The single real invariant form on $\mathcal{L}SO(3)$ becomes a single dual invariant form on $\mathcal{L}E(3)$, the dual dot product. Hence there are just two <u>real</u> invariant forms on $\mathcal{L}E(3)$, the scalar and dual parts of the dual dot product. Rather these two forms give a basis for the 2-dimensional vector space of invariant forms on $\mathcal{L}E(3)$. The scalar part of the dual dot product is a multiple of the Killing form of E(3), degenerate since E(3) is not semi-simple.

That is, if $r = (a_1V_1+a_2V_2a_3V_3)+ \varepsilon(b_1V_1+b_2V_2+b_3V_3) \in \mathcal{L}E(3)$

then the two forms are

$$q_1(r,r') = a_1a_1' + a_2a_2' + a_3a_3'$$

$$q_2(r,r') = a_1b_1' + a_2b_2' + a_3b_3' + a_1'b_1 + a_2'b_2 + a_3'b_3$$

The pitch $p(r)$, of motor r is usually given as the ratio

$$p(r) = q_2(r,r) / 2q_1(r,r)$$

see [1] for example. However this suggests that for a motor with $q_1(r,r) = 0$, the pitch is undefined. We prefer to think of the pitch as a rational map from $\mathbb{P}\mathcal{L}E(3)$, the space of screws to $\mathbb{P}\mathbb{R}^1$, given in homogeneous co-ordinates by

$$p(r) \longmapsto (q_2(r,r),\ 2q_1(r,r))$$

Now screws for which $q_2(r,r) = 0$ and $q_1(r,r) = 0$ are in the closure of this map and their image is clearly the point $(1,0)$. The pre-image of the point $(0,1)$, the zero pitch screws; satisfy $q_2(r,r) = 0$. This is the familiar Klein quadric of lines in $\mathbb{P}\mathbb{R}^3$, recall that $\mathcal{L}E(3)$ is a six dimensional vector space so $\mathbb{P}\mathcal{L}E(3)$ is isomorphic to $\mathbb{P}\mathbb{R}^5$.

The proof that there are just two invariants on the space of screws is new. It has deep implications for the formulae one can use to describe physical screws and screw systems. The group of proper Euclidean motions $SO(3) \circledS \mathbb{R}^3$, is clearly of central importance for spatial kinematics. This was certainly Study's view and we feel sure that a re-examination of his work in the light of modern developments in group theory and geometry would pay handsom dividends.

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