NOTES & RECORDS OF THE ROYAL SOCIETY

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Notes Rec. R. Soc. 2005 **59**, doi: 10.1098/rsnr.2005.0097, published September 22, 2005

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Notes Rec. R. Soc. (2005) **59**, 285–304
doi:10.1098/rsnr.2005.0097
Published online 22 September 2005

ARTHUR CAYLEY FRS AND THE FOUR-COLOUR MAP PROBLEM

by

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The four-colour map problem (to *prove* that on any map only four colours are needed to separate countries) is celebrated in mathematics. It resisted the attempts of able mathematicians for over a century and when it was successfully proved in 1976 the 'computer proof' was controversial: it did not allow scrutiny in the conventional way. At the height of his influence in 1878, Arthur Cayley had drawn attention to the problem at a meeting of the London Mathematical Society and it was duly 'announced' in print. He made a short contribution himself and he encouraged the young A. B. Kempe to publish a paper on the subject. Though ultimately unsuccessful, the work of Cayley and Kempe in the late 1870s brought valuable insights. Using previously unpublished historical sources, of letters and manuscripts, this article attempts to piece together Cayley's contribution against the backcloth of his other deliberations. Francis Galton is revealed as the 'go-between' in suggesting Cayley publish his observations in *Proceedings of the Royal Geographical Society*. Of particular interest is that Cayley submitted *two* manuscripts prior to publication. A detailed comparison of these initial and final manuscripts in this article sheds new light on the early history of this great problem.

Keywords: four-colour problem; graph theory; Galton; Kempe; topology; proof

Introduction

The question of whether four colours are sufficient to separate countries on any map has been of keen interest to mathematicians for more than 150 years.[1] In 1977, Thomas Saaty and Paul Kainen wrote of its fame: '[t]he four-color conjecture has been one of the great unsolved problems of mathematics. From 1852 and continuing to this day, practically every mathematician who has ever lived has, at one time or another, tried his or her hand at settling the conjecture.'[2] This indicates the important place that intractable problems hold in the life of mathematics, particularly when they are capable of arresting description. Coinciding with the publication of Saaty and Kainen's book, the four-colour map problem was finally solved. The new theorem was proved, or rather 'proved', because, in harnessing modern computing power as an essential ingredient in its demonstration, the methodology of the proof is still considered

Figure 1. Arthur Cayley FRS (date unknown). (Photographer Maull & Polyblank. Copyright © the Royal Society.)

contentious in some quarters. This and the further problems it spawned makes the four-colour map problem of continuing interest.

As with the many conjectures and mathematical themes that emerged in the nineteenth century, Arthur Cayley FRS (1821–1895; figure 1) played a leading part in its development.[3] Using unpublished sources, I will outline the main threads of his contribution to the four-colour map problem, and sketch his mathematical activity between 1877 and 1882, when the problem was highlighted and brought before the mathematical community. Of special interest are the roles of Francis Galton FRS (1822–1911) and Alfred Bray Kempe FRS (1849–1922). It will be seen that Cayley's encouragement of Kempe's work was a decisive element in the publication of the young man's contribution to the solution of the problem. In regard to his own work, it is little known that Cayley prepared *two* manuscripts before the publication of his paper on the subject, perhaps prompted by Galton.[4]

Although the four-colour problem had been known to a few mathematicians for about 20 years, Cayley's announcement in June 1878 opened the challenge to a wider group. His query on whether a solution to the problem had been given was published in the *Proceedings of the London Mathematical Society* and in the popular science journal *Nature*. By this, according to Saaty and Kainen, the four-colour problem was formally launched, 'its colorful career involving a number of equivalent variations, conjectures, and false proofs. But the question of sufficiency, which had been wide open for such a long time, is now [1977] closed.'[5]

Cayley's pathway

Why was Cayley attracted to the problem? One aspect lay in his attitude to 'colour.' In the years 1877–79, in particular, Cayley exploited the uses of colour in mathematics, something not usually associated with the subject. His water-colour sketches taken into the lecture room could show off polyhedra to advantage, but he also found colour useful in the investigative process itself. His attempt to construct a theory of colour (based on linear equations) came to little, but his graphical group theory, in which he introduced the notion of the graph of a group, which he later termed a colourgroup (*Gruppenbild*, now called a 'Cayley colour graph'), suggested a geometric approach to group theory that has proved fruitful.[6] In his hands the Newton–Fourier method of mathematical analysis also made use of colour. Originally motivated by the practical root-finding of equations, the method was extended to functions of a complex variable by Cayley, and his work can now be interpreted as research in fractals. Here he was in a 'grey-scale' phase, in which regions of the complex plane were labelled white, grey and black. During these years, he also reconsidered Cauchy's work on equations and this time developed a broader palette. A geometric perspective to this theory, coupled with the complexity of the visual evidence, made the use of colour desirable, and he spoke of red and blue curves, and regions tinted in sable, gules, argent and azure.[7] For Cayley, colour was much more than a cosmetic device, for it could be used for achieving clarity, making new discoveries and suggesting valuable ideas.[8]

The four-colour map problem also has links with the theory of polyhedra, and Cayley had a lifelong interest in this subject. In 1858, in the same month as he presented his famous Sixth Memoir on quantics to the Royal Society (in which he showed that Euclidean geometry was subsumed in the wider notion of non-Euclidean geometry),[9] he published a paper on polyhedra. In this he extended Euler's basic $v+f=e+2$ formula (relating the numbers of vertices (v), faces (f) and edges (e) of a polyhedron) to apply to polyhedra that were regular in a less restrictive way than required by the five regular convex Greek solids (the Platonic solids). Since the time of Euclid, four new 'Kepler–Poinsot' regular non-convex star polyhedra were added to the Platonic solids. Johannes Kepler found two distinct dodecahedra with 12 star-shaped faces, and in 1809 Louis Poinsot added two other polyhedra that have star-shaped vertices. Cayley called Kepler's specimens the 'small stellated dodecahedron' and the 'great stellated dodecahedron' and Poinsot's the 'great dodecahedron' (with 12 faces) and the 'great icosahedron' (with 20 faces). Euler's formula fails for two of these nine regular polyhedra. Cayley's tack was to generalize Euler's formula to one of the form $mv+nf=e+2D$ (in which D is the 'density' of a polyhedron, a 'covering number' calculated by projecting the polyhedron onto a concentric sphere placed inside it). This generalized formula applies without exception to the five Platonic solids and the four Kepler–Poinsot polyhedra.[10]

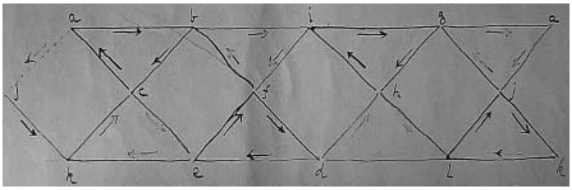


Figure 2. Diagram for Cayley's hemihedron as it appears in manuscript (1878). (Reproduced courtesy of Niedersächsische Staats- und Universitätsbibliothek, Göttingen.)

More on the subject of polyhedra was to follow from Cayley, and he studied topology from a more abstract point of view in his 'Partitions of a close' in 1861.[11] This has a bearing on his study of the four-colour map problem, but as we shall see this is made clear only by the existence of the preliminary draft for his printed paper 'On the colouring of maps' (1879). Cayley defined a 'close' as a subspace that is path connected. An analytic definition was not forthcoming but it was defined intuitively as an 'enclosed space, such that no part of it is shut out from any other part of it, or, what is the same thing, such that any part can be joined with any other part by a line not cutting the boundary, is termed a *close*'. His notion of a 'path' was defined in terms of drawing, and although his definition would hardly pass an exacting test of rigour, it illustrates his perspective as an intuitionist mathematician: 'the words line and curve are used indifferently to denote any path which can be described *currente calamo* [with the pen running on; his italics] without lifting the pen from the paper'.[12] In letters to his friend J. J. Sylvester, Cayley explored the notion of path-connected regions and the topology of the plane.[13]

Once again we see Cayley quickly grasping an important mathematical idea. In the company of J. B. Listing, a student of Gauss's who introduced the word *Topologie*, and Camille Jordan, Cayley recognized the topological character of polyhedral formulae of the type $v+f=e+2$, namely that their truth does not depend on edges being 'straight' or faces being 'flat'.[14] The four-colour map problem is also mathematically related to the enumeration of chemical 'trees', an area of interest for Cayley in the mid-1870 s. These various connections suggest that the four-colour map problem was situated in a unified orbit of ideas that absorbed him, and was not an isolated one-off problem.

The London Mathematical Society

In the late 1870 s, Cayley's visits to London were made to attend meetings at the Royal Society, the Royal Astronomical Society, and the London Mathematical Society. On 13 June 1878 he attended a committee meeting of the London Mathematical Society in the early evening.[15]

Business was first attended to. Some papers presented to the previous meeting (9 May) were passed for publication in the *Proceedings*. At their meeting in May, Olaus Henrici had read a paper submitted by Felix Klein, 'On the transformation of elliptic functions', and A. B. Kempe read his paper on 'conjugate four-piece linkages'. Cayley had read a paper on group theory, and it was while writing up his paper for publication in the *Proceedings* that his mind turned to the use of colour as an aid to understanding. He had written to Klein:

> I have to thank you very much for your letter, and I had also the pleasure of hearing the paper [On the Transformation of Elliptic Functions] read at the London Math. Society.

> I am very glad that you have given us something for our Proceedings .... I have just written out for the L.M.S. a short paper in Groups, containing an idea which seems to me promising. In a substitution group of the order n upon n letters, every substitution must be *regular* [having cycles of equal length]; Taking $m=12$, & considering the 12 letters *abcdefghijkl* suppose that the substitution of the group is *abc.def.ghi.jkl ... the diagram has the property that any such symbol* [representing operations on the letters] *which leads from any one letter to itself, leads also from every other letter to itself ...*[16]

Cayley went on to describe his colour diagrams. In the case of the group of order 12, the appropriate diagram was obtained from an octo-hexahedron with 8 triangular faces (4 red, 4 black) and 6 square faces (white, say).[17] In the construction of the hemihedron (figure 2), formed by the removal of some faces, he wrote: 'the essential characters [of these group diagrams] are that the lines of any given colour shall form polygons of the same number of sides ... that there shall be from each point only one line of the same colour; ... we can if we please, introduce into the diagrams a set of lines of a new colour to represent any dependent substitution of the group.'[18]

In modern terms the colours of the octo-hexahedron amount to a colouring of the polyhedron in which no two adjacent faces are coloured with the same colour. This is a problem mathematically connected to the four-colour problem.

The June committee meeting of the London Mathematical Society was followed by the full meeting of the Society. This took place at 22 Albemarle Street, Piccadilly, in the rooms of the Royal Asiatic Society in the fashionable West End of London, where meetings began at 8 p.m. and lasted for about two hours.[19] The society met monthly and this June meeting would be the last of the academic year. The small gathering comprised the committee, which had met earlier, plus a few extra members.[20]

It was a busy meeting. J. J. Walker gave an account of his analysis of plane curves, and R. Tucker contributed a paper sent in by Hugh MacColl (41 years old) on logic, and one by Cecil James Monro (44) on geometry. Monro was Cambridge trained and wrote variously on probability, logic and geometry and physics; MacColl was a graduate of the University of London and earned his living as a teacher in Boulogne-sur-Mer.[21] Papers written by Samuel Roberts and Robert Rawson were also read, and T. A. Hirst gave an account of George-Henri Halphen's theory of the characteristics of conics. Cayley did not much care for Halphen's work, but when a subject attracted him he could be relied upon to make a comment, and he duly appended a note to Monro's paper.

Only the day before the meeting, Cayley submitted his tenth memoir on quantics to the Royal Society, the final part of his long-running series in invariant theory. He could afford to survey the mathematical scene and involve himself with other projects. In the minutes it was recorded that 'Questions were asked by Prof. Cayley, Mr Merrifield and Mr Tucker.'[22] These questions were amplified in the pages of *Nature* and the *Proceedings* that carried a report of the meeting, including a 'call to arms' for mathematicians to meet the challenge posed by the four-colour map problem:

> Questions were asked by Prof. Cayley, F.R.S. (Has a solution been given of the statement that in colouring a map of a country, divided into counties, only four distinct colours are required, so that no two adjacent counties should be painted in the same colour?); by Mr. Merrifield F.R.S. on "the uniform Distribution of Points in Space"; by Mr Tucker, in connection with ... a second exception to Fermat's statement "that all numbers of the form $2^{2^n}+1$ are primes."[23]

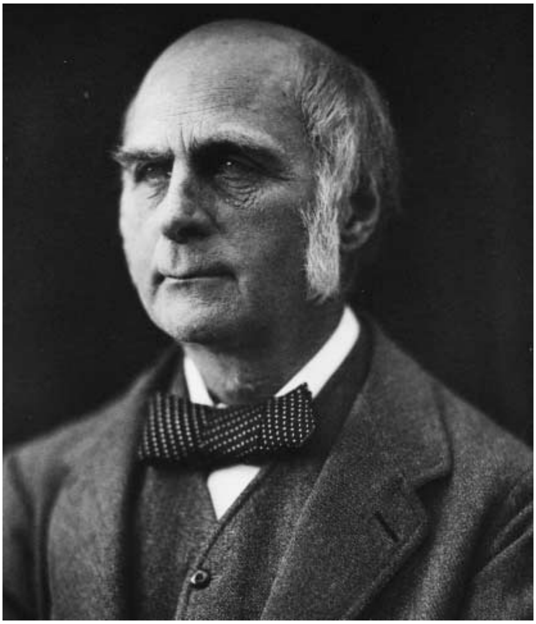


Figure 3. Sir Francis Galton FRS (date unknown). (Copyright © the Royal Society.)

The Royal Geographical Society

Situated on the other side of Hyde Park from the London Mathematical Society headquarters, the Royal Geographical Society in Kensington Gore was about to launch a new series of their *Proceedings*. Francis Galton (figure 3) had heard about the map-colouring problem—he was an avid reader of journals and newspapers, and his collection of newspaper cuttings was legion.

Living at 42 Rutland Gate, Knightsbridge, Galton was a gentleman of science who pursued his interests unhindered by the need to earn a living by paid employment. What made him ideal in the role of disseminator of ideas was his broad knowledge of many things, a latter-day William Whewell. Whereas Cayley could claim depth (but this was the very last thing he would do), Galton could claim breadth. From an early age, Galton's background and training suggested a polymath in the making. Widely travelled as a youth, his early studies in medicine were followed by Cambridge mathematics. At the age of 30 years it was claimed that '[h]is experience had been such that he knew more of mathematics and physics than nine biologists out of ten, more of biology than nineteen mathematicians out of twenty, and more of pathology and physiology than forty-nine out of fifty of the biologists and mathematicians of his day.'[24] As a geographer, the four-colour map problem did not place practical difficulties in Galton's way and he firmly believed that four colours would suffice for any map. The problem itself was perhaps immaterial to Galton, but the fact that it interested mathematicians, and Cayley in particular, suggested that there was publicity value in it—Galton was a gadfly in the Victorian scientific community. Here was an opportunity to use Cayley's scientific reputation to help launch the new series of the *Proceedings*.

Galton was a friend of Cayley's; they had been together at Cambridge in the 1840 s. Cayley emerged from the Tripos competition as Senior Wrangler of his year (1842), but although Galton had started with high hopes he graduated with an ordinary degree in 1843. One year older, Cayley was one of Galton's early heroes, a view that remained undimmed as the years passed. 'Never was a man whose outer physique so belied his powers as that of Cayley', he wrote in his autobiography, remembering a student reading party in Scotland supervised by Cayley. 'There was something eerie and uncanny in his ways', he wrote, 'that inclined strangers to pronounce him neither to be wholly sane nor gifted with much intelligence, which was the very reverse of the truth.'[25] In recalling his frail appearance, Galton observed Cayley's physical prowess: '[o]ne morning he coached us as usual and dined early with us at our usual hour. The next morning he did the same, all just as before, but it afterwards transpired that he had not been to bed at all in the meantime, but had tramped all night through[,] over the moors to and about Loch Rannoch.'[26] Galton recognized a kindred spirit: like himself, Cayley was an explorer of the wider intellectual terrain, a man who pressed ahead with projects without the inclination to dwell.

After Cambridge their professional lives had diverged. Cayley went in for the law, but after his graduation an inheritance absolved Galton from the need of paid employment. They remained in touch and Cayley was called on for mathematical advice from time to time. In the 1860 s, Cayley was a supporter of *The Reader*, a short-lived weekly journal that Galton promoted. Whereas Cayley was a dedicated mathematician who generally stayed within the boundaries of the subject (although broadly conceived), Galton was a generalist who declared one of his specialisms at the outset of his scientific career: 'I am a traveller by inclination and my study is geography.'[27] In this subject, his object was to 'revolutionise and humanise maps'.[28] A leading interest for us is Galton's use of colour in map construction, thus taking advantage of the technology of colour lithography then being advanced. One of his ideas was to designate natural areas using colour coding, as for instance with his 'tawny plains' and 'russet forests'. In 1872 he proposed to the council of the Royal Geographical Society that money be spent 'in procuring specimens of, and a report on, the various styles of cartographic representation now in use both in England and abroad, as regards shading, colours, symbols, and method and cost of production.'[29]

In the summer of 1878, Galton went on vacation to Vichy in France. Cayley's mathematical confidante J. J. Sylvester FRS, who had been appointed as professor of mathematics at Johns Hopkins University two years before, came back to Europe for the summer as was his custom, and stayed with Cayley in early July. He tried to persuade Cayley to attend the Dublin meeting of the British Association, but instead Cayley went off for his summer to the Lake District, and stayed at Grasmere.[30] At some point Galton approached him with a request that he write an article on the four-colour map problem for the journal of the Royal Geographical Society. Cayley may have taken Galton's request with him to the Lakes, but in any event he accepted Galton's invitation to write on the problem.

With Cayley's return to Cambridge in the autumn, worldly matters had to be weighed. Despite our view of the leisurely life of a nineteenth-century Cambridge professor, there were pressures. In the university a high-profile committee on the reform of the Tripos was taking place, and Cayley's income, which depended on the rents from farms in Hampshire was under threat at the height of an agricultural depression. In mathematics he gave his undergraduate lectures on solid geometry, and though to small numbers, they had to be given.[31] Evidently they were the spur for a new paper on polyhedra in which he returned to the ideas of projecting a polyhedron onto a concentric sphere contained within it, and of the density of a polyhedron.[32] Indeed, his enquiry into the state of the four-colour problem in June may have guided him to choose the subject of solid geometry as the topic for his lectures of 1878. He had free rein in these choices and frequently used them as a launch pad for his research. His lectures on polyhedra brought him into contact with the four-colour map problem once again, for it is a special case of the problem of colouring the faces of a polyhedron so as to satisfy specified requirements.[33] Thus, in the latter half of 1878 Cayley was involved with geometrical ideas that impinged on this question in graph theory. At some point in this period he sketched out some first thoughts on the four-colour map problem, and discovered that the general question was reducible to one for trivalent graphs.

Of especial note is that Cayley submitted two manuscripts to the editor of the journal, the one he used as a basis for the published paper, and also a preliminary one. The initial manuscript reveals that Galton was the go-between connecting the austere world of mathematics and abstract truth with the practical world of geography where intellectual appetites were satisfied by explorers' tales brought home from the lands of the Empire.

The new year

On receipt of a manuscript from Cayley, Galton wrote to the editor of the *Proceedings of the Royal Geographical Society*, Henry Walter Bates.[34] Bates had become assistant secretary of the Royal Geographical Society in 1864, serving in that capacity for almost 30 years, and he edited publications of the Society. About Cayley's submission, Galton informed Bates:

> I wrote to Prof. Cayley of Cambridge, asking if he could send a short paragraph suitable for the Proceedings explaining as far as feasible the very curious problem of the sufficiency of 4 colours to distinguish adjacent territories in maps, which has lately excited great interest among the higher mathematicians.
>
> The problem is especially associated with Prof. Cayley's name & simple as it may be thought at first sight, it still baffles the highest analysis though there is no doubting its truth.
>
> I enclose his reply. It is much longer & more technical than I had hoped, but it makes the character of the problem very clear & I am sure it would interest a certain small section of our society. It is also something to obtain a paper from the most eminent mathematician in England, and probably in the world, on a geographical topic.
>
> Therefore it ought I think to be submitted to Council for reference. If it should happen to be referred to myself I should recommend it for publication either in the Proceedings or in the Journal according to the discretion of the Editor.
>
> You see that the author wishes to have a proof.[35]

'Proof' was indeed what Cayley wished for, though not the proof-sheets Galton had in mind.

Galton's reservation, that Cayley's initial manuscript was 'much longer & more technical' than he had hoped, may have encouraged the submission of the second manuscript.[36] Cayley was sensitive to the needs of non-mathematical readers and attempted to reduce the level of mathematics, although stepping down was difficult for him. The second manuscript is moderately less technical, and although it is no shorter it represents a radical revision of the material initially presented to the Royal Geographical Society in the first draft.

CAYLEY'S TWO MANUSCRIPTS*

Preliminary manuscript:
Note on the Colouring of Maps
by Professor Cayley.

It is mentioned somewhere by the late Prof. De Morgan, as a theorem known to map-makers, that four colours are sufficient for any map.

Final manuscript:
On the Colouring of Maps.
by Professor Cayley.

The theorem that four colours are sufficient for any map, is mentioned somewhere by the late Professor De Morgan, who refers to it as a theorem known to map-makers.

> The 'somewhere' has since been identified as in the *Athenaeum*.[37]

To state the theorem in a precise form, let the term "close" be used to denote a ~~tract of country such that~~ plane area such that every point of it is, without passing out of the area, accessible from every other point of it: for instance the area within any closed curve such as a circle is a close; and so also, if the circle including within it a circle or two or more detached (or non-intersecting) circles, then the area, exclusive of the areas within the included circle or circles, is a close; but of course the area within any two or more of the detached circles is not a close; only the area within any one of these is a close by itself. And further the area within any self-intersecting curve such as a figure of eight is not a close, but the areas within the different loops thereof are distinct closes a close (see back).[a]

[a] [A close] ~~is in fact~~ is in fact an area which is simply or multiply connected (einfach or mehrfach zusammenhangend).

To state the theorem in a precise form, let the term "area" be understood to mean a simply or multiply connected area:[b]

> In this final manuscript Cayley substituted 'area' for the more technical 'close' and 'attached' for his original 'coterminal'. He used 'appointed' for areas meeting at point(s). P. G. Tait used 'coterminous' in his own deliberations.
> ←

[b] An area is "connected" when every two points of the area can be joined by a continuous line lying wholly within the area; the area within a non-intersecting closed curve, or say an area having a single boundary, is "simply connected"; but if besides the exterior boundary there is one or more than one interior boundary (that is, if there is within the exterior boundary one or more than one enclave not belonging to the area) then the area is "multiply connected": the theorem extends to multiply connected areas, but there is no real loss of generality in taking, ~~them to be~~ and we may for convenience take, the areas of the theorem to be each of them a ~~simp~~ simply connected area.

* The manuscripts are held at the Royal Geographical Society (JMS 19/24, Maps 1879). Both are undated, but in choosing one to be the preliminary manuscript, and one to be the final, I have relied on Galton's remarks and the fact that the printed paper is derived from the 'final' manuscript. This second manuscript may have been written by Cayley as a reaction to Galton's comment that the initial one 'is much longer & more technical than I had hoped'. My editorial notes are placed in bordered boxes; the symbol // indicates a page break in the manuscript; Cayley's intended footnotes have been labelled a, b and c. Cayley's deletions have been retained but his minor revisional insertions have been made silently.

Suppose also that two closes which touch each other along a boundary-line are said to be "coterminal", it being understood that two closes touching each other at a point or points, are not coterminal; for instance if a circle be divided by two diameters into four quadrants then two ~~then~~ adjacent quadrants ~~are~~ coterminal but ~~then~~ two opposite quadrants are not coterminal.

The theorem then is that ~~if~~ dividing in any manner whatever a plane area into closes it is possible with four colours only, to colour these in such wise that two coterminal closes have never the same colour.

This portion {…} is not present in the final manuscript, but was added to the text of the Note at a later stage. It appears in the printed version. →

It is clear that four colours are wanted; for instance if we have a circle divided into ~~the~~ three sectors,[c] then the first of these is coterminal with the second, the second with the third and the third with the first; they require therefore three colours say red, blue, green: hence if the whole circle be included within another circle, the close formed by the annulus must have a fourth colour say yellow.

If instead // of three, there had been four sectors, then for these two colours would have been sufficient; they might be red, blue, red, blue, alternately; and so in general for any even number of sectors two colours are sufficient, but for any odd number three are required. But in

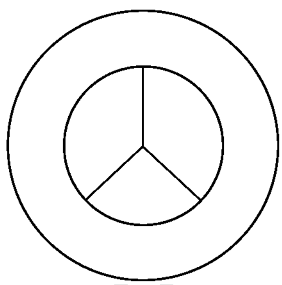
[c] In connexion with what follows, it would be perhaps convenient to use instead of sector the word segment; it being understood that as regards a closed figure like the circle segment means sector, viz. that division is made by lines drawn from a common point to the boundary.

and let two areas, if they touch along a line, be said to be "attached" to each other; but if they touch only at a point or points, let them be said to be "appointed" to each other. For instance if a circular area be divided by radii into sectors, then each sector is attached to the two contiguous sectors, but it is appointed to the several other sectors.

The theorem then is, that if an area be partitioned in any manner into areas, these can be, with four colours only, coloured in such wise that in every case ~~no~~ two attached areas have distinct colours: appointed areas may have the same colour.

{Detached areas may in a map represent parts of the same country, but this relation is not in anywise attended to; the colours of such detached areas will be the same, or different, as the theorem may require.}

It is easy to see that four colours are wanted; for instance, we have a circle divided into three sectors, the whole circle forming an enclave in another area, then we require three colours for the three sectors, and a fourth colour for the surrounding area:

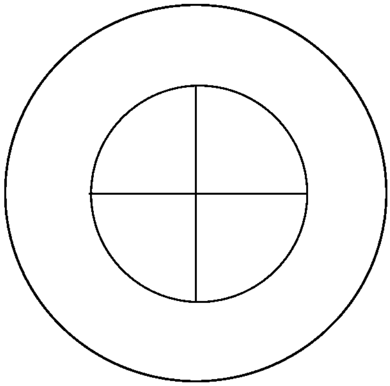


This map requires four colours

either case the annulus being coloured, say yellow, then the three colours, red, blue, green, are again at liberty to be used for any close or closes outside and coterminal with the annulus; in particular suppose that surrounding the ~~before-m[entioned] we see in this way generally how the theorem may be true; and also in a similar manner the difficulty that there is in a strict and complete proof of it: for instance let the figure comprise as before a circle divided into three sectors, and a surrounding annulus, and surrounding this a second annulus (say the inner annulus): & also surrounding this a second or outer any annulus, divided suppose into three portions by lines connecting its two boundaries~~ before mentioned (or say the inner) annulus, we have a second or outer annulus itself divided (by lines connecting its two boundaries) into an odd or an even number of parts (or say ~~sectors~~ segments) then for colouring these we have available the three colours used (or only two of them used) for colouring the sectors of the circle.

We thus see in a general way how the theorem may be true; but we see also the difficulty that there is in giving a strict and complete proof of it: for instance let the circle be divided into 3 sectors, and the outer annulus into 3 ~~sectors~~ segments: if the inner annulus be undivided, this is the case already considered; but let the inner annulus be divided suppose into ~~two~~ 2 ~~sectors~~ segments. There will be one of these coterminal with all the sectors of the circle and (these being blue green & red) it must be yellow; the other of them will then be coterminal with two or only one of the sectors of the circle, in the former case the colour will be determined, in the latter case it will be one at pleasure of two colours; say in either case the colour is red, viz. the two segments of the inner annulus are yellow & red: it is then to be

then for these two colours would be sufficient, and taking a third colour ~~four~~ for the surrounding area, three colours only would be wanted; and so in general according // as the number of sectors is even or odd, three colours or four colours are wanted.



This map requires only three colours

And in any tolerably simple case it can ~~in general~~ be seen that four colours are sufficient.

Cayley apparently sees ‘segment’ as more general than ‘sector’ as it does not imply that the ‘areas’ need be circular in shape.

←

Cayley mentions *specific* colours, red, blue, green, yellow, only in the *initial* manuscript, but not in the final manuscript.

shown that however the three segments of the outer annulus are situate with regard to the segments of the inner annulus, it is possible to colour them with the four colours: this it would be easy to do; but the number of the // segments of the inner annulus (instead of 2) might have been 3, 4, 5.. or any other number; and the question is to obtain a general proof: this I have never succeeded in finding.

But I have not succeeded in obtaining a general proof:

> In the final manuscript, Cayley explains the difficulty with using the method of mathematical induction to prove the four-colour theorem.
> →

and it is worth while to explain wherein the difficulty consists. Supposing a system of *n* areas coloured according to the theorem with four colours only, if we add an (*n*+1)th area, it by no means follows that we can without altering the original colouring, colour this with one of the four colours. For instance, if the original colouring be such that the four colours all present themselves in the exterior boundary of the n areas, and if the new area be ~~any~~ an area enclosing the n areas, then there is not any one of the four colours available for the new area.

> A proof is required for *any* maps, but Cayley's 'patching' technique demonstrates it is necessary to consider only 'cubic maps'—ones in which exactly three countries meet at a point.
> Stated in modern terms, it is necessary to consider only graphs that are *trivalent*.
> →

The theorem, if it is true at all, is true under more stringent conditions; for instance ~~imagine four or if a~~ if in any case the figure includes four or more areas meeting in a point (such as the sectors of a circle), then if (introducing a new area), we place at the point a small circular area, cut out from and attaching itself to each of the original sectorial areas, ~~then~~ it must according to the theorem be possible with four colours only, to colour the new figure; and this implies that it must be possible to colour the original figure so that ~~not more~~ only three colours (or it may be two) are used for the sectorial areas. And in precisely the same way (the theorem is in fact really the same) it must be possible to colour the original figure in such wise that only three colours (or it may be two) present themselves in the exterior boundary of the ~~original~~ figure.

> A 'limited proof' may be obtained by assuming that countries on the boundary require at most three colours for their separation. But even in this case there are difficulties with the mathematical induction.
>
> →

But ~~if the~~ now suppose that the theorem under these more stringent conditions is true for n areas: say that it is possible with four colours only, ~~and in the~~ to colour the n areas in such wise that not more than three colours present themselves in the external boundary: then it might be // easy ~~enough~~ to prove that the n+1 areas could be coloured with four colours only: but ~~we~~ this would be insufficient for the purpose of a general proof; it would be necessary to show further that the n+1 areas could be with the four colours only coloured in accordance with the foregoing boundary condition; for without this we cannot from the case of the n+1 areas pass to the next case of n+2 areas. And so in general, whatever more stringent conditions we import into the theorem as regards the n areas, it is necessary to show not only that the n+1 areas can be coloured with four colours only, but that they can be coloured in accordance with the more stringent conditions. As already mentioned, I have failed to obtain a proof.

Mr Galton has suggested to me the analogous problem as regards three dimensional space. A close would mean a simply or multiply connected portion of space: closes touching ~~over a surface close or surface closes being at only at a point or points or along~~ over a surface-close or surface- closes would be coterminal, but those touching only in a point or points, or along a line or lines, would not be coterminal; and then ~~question is how many colours &~~ dividing in any manner space (or a ~~space~~ portion of space) into closes, the question is how many colours are required in order to colour these in such wise that ~~no~~ two coterminal closes have never the same colour. The ~~determination~~ discovery of the number would be perhaps easy; but the demonstration of ~~the~~ its sufficiency ~~of the number~~ would certainly be very difficult~~y~~: the complexity is greatly increased by the ~~circumstances~~ necessity that there would be of examining cases where ~~the~~ a surface-close ~~or surface closes~~ common to two coterminal closes is a multiply-connected surface-close.

> This section on a three-dimensional version of the four-colour theorem is omitted in the final manuscript and in the printed version.
>
> ←

Cayley had made important observations about the nature of the problem and introduced the patching technique but, as he openly admitted, he had not been able to prove the general theorem. At their monthly meeting in February 1879, members of the Royal Geographical Society heard about the geographical exploration of Afghanistan and India. Written up in their report, which appeared in April, there was also Cayley's paper 'On the colouring of maps'. This was essentially the content of the second manuscript.

By then, Cayley had moved on. He had discovered a facet of the 'Newton–Fourier' root-finding algorithm that demanded exploration. As a first step, he set a question on it for the Smith's Prize Examination at Cambridge University in January 1879.[38] In February he became quite excited about this problem, writing enthusiastically to T. A. Hirst and William Thomson about it and, as we have noted, involving the use of colour to explain it.[39] He gave a talk on it to the Cambridge Philosophical Society and after this wrote to Sylvester:

> I send herewith a problem which has interested and bothered me a great deal—tho' I now in some measure see my way to a solution. I think it will please you, and I shall be much obliged for any new lights you can give me on the subject.[40]

Cayley's *modus operandi* was to make a contribution, publish it, and to move on to pastures new. The Newton–Fourier problem was the new enthusiasm and the four-colour map problem was pushed into the background.

Kempe's proof

A year after Cayley publicly queried the progress on the four-colour map problem, and a few months after his own partial solution of the problem had been published in the Royal Geographical Society's *Proceedings*, he received a letter from A. B. Kempe, a former student at Cambridge. Kempe had attended the London Mathematical Society meeting on 13 June 1878 when Cayley had raised the matter.

Alfred Bray Kempe (figure 4) attended Trinity College, Cambridge, and had graduated as twenty-second Wrangler in 1872. Much to the consternation of his friends, his lowly place in the Wrangler list was nothing short of an injustice. Kempe was bolstered by their support, but the failure to attain a high position in the Tripos order of merit may have contributed to his seeking a legal career rather than an academic position. He became a barrister, but he kept up with mathematics and contact with Cambridge through his friend J. W. L. Glaisher. He became the gentleman of science in the Cayley mould, 'an excellent man of business, with a sound judgment, great tact, and endless good nature', later becoming Treasurer of the Royal Society.[41]

One of Kempe's earliest papers in mathematics was on the solution of polynomials by a mechanical method. Afterwards he published a number of papers on linkwork, a topic of interest to mathematicians of the 1870 s, and the paper that he read to the May meeting of the London Mathematical Society in 1878 was one that had interested Cayley. Flushed with success with his forays in linkwork, Kempe rose to the challenge of the four-colour problem. In its dual form, Kempe actually saw the problem as a problem in linkwork: a piece of tracing paper was placed over the map, and, marking each country with a cross, the crosses are joined by links across boundaries. After a short time, Kempe believed he solved the problem that Cayley had posed. Having made such progress it was natural for him to submit his solution to his former tutor and Cayley responded with encouragement:

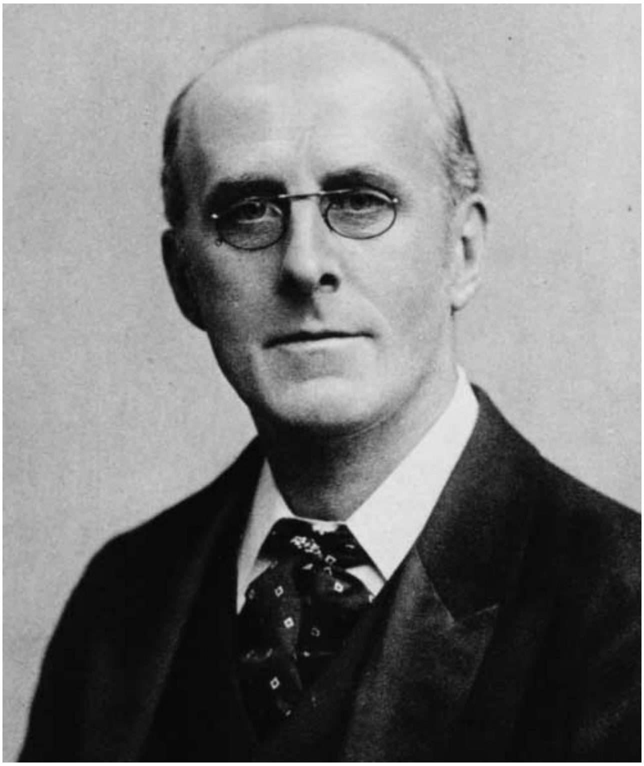


Figure 4. Sir Alfred Bray Kempe FRS (date unknown). (Copyright © the Royal Society.)

> The solution seems to me quite correct, & I am delighted with it. I think it might very well go, in nearly its present form to the Geographical Magazine, and I trust you will send it— I will write to Mr Bates to say that I have seen it ….
> Geometrically, I think the correlative [dual] form of the theorem is more simple (tho' less adapted for the Geographical Magazine) viz. "Considering on a sphere any number of points joined together in any manner by lines which do not cross each other then it is possible with four letters only, to mark the points, so that in each of the joining lines, the two extremities shall be marked by different letters." Your proof I think applies directly, and even more easily to this form of the theorem….
> Is the theorem true where the surface is a multiply connected surface such as the anchor ring [a torus]—I think then that at each point there might be 6 or more lines: and if so the proof would fail.[42]

Kempe produced a shortened version of his paper and published it in the *Proceedings of the London Mathematical Society*. With this publication both Kempe and Cayley firmly believed the 'problem' had become a 'theorem'. As Kempe had confidently written:

> The theorem is this:—The districts into which any map is divided by its boundaries can be coloured with four colours, so that no two adjacent districts (*i.e.* districts which have one or more common boundaries, as distinguished from those which are remote from each other, or touch only at detached points) may be coloured the same colour.[43]

A short announcement also appeared in the pages of *Nature*.[44] Evidently Kempe's technical paper was unsuitable for members of the Royal Geographical Society. If Cayley's prose-styled paper had proved too mathematical for members, Kempe's technical proof would have left a lesser impression. It is tempting to speculate that Galton deflected Kempe's proof from appearing in the *Proceedings*, for one can easily imagine him believing that members had been sufficiently enlightened by Cayley's effort. Moreover there would be no merit to be gained in publishing a paper written by a tyro. Instead, Kempe submitted his paper for publication in the *American Mathematical Journal*, a journal founded by J. J. Sylvester several years earlier.[45] In the summer of 1879, Sylvester wrote to him:

> I duly received your valuable manuscript and ought to have acknowledged the receipt of it at once—How I failed to do so I can hardly understand. It is now on its way to America and I hope will be in print before 3 months are over—You will receive proofs from the other side.[46]

By this time, Sylvester would have known of Cayley's acceptance of Kempe's proof, and this would have been added justification for him to accept it for his own journal.[47]

Kempe's proof of the four-colour problem contributed to his scientific standing. With support from the leading mathematicians, he was clearly in line for elevation to membership of the Royal Society. Sylvester, who revelled in Royal Society politics, recommended that Oxford's Henry Smith lead the application: 'I mention the name of Henry Smith in especial [*sic*] because he is a more practical man than Cayley, who I am sure would be among your particular well wishers and do all in his power to promote your election.'[48] In his promoting of Kempe, Cayley was only troubled by his acute sense of propriety. He wrote to Kempe in the next month: 'I shall have much pleasure in signing your [membership] certificate: please send it soon, as I ought to do so *before* I am (as I suppose I shall be) a member of the [Royal Society] Council.'[49] He proposed Kempe two days later, on 24 November 1879, and was the first to sign the membership certificate, but Kempe had to wait and was elected on 2 June 1881. The citation cautiously declared his proof as 'solving, it is believed for the first time, the problem of map delimitation with four colours.'[50] Published in the second volume of Sylvester's *American Journal*, his proof survived for 10 years before it was shown to be deficient by P. J. Heawood.[51]

Although of considerable interest, the four-colour map problem was just one of Cayley's deliberations at this time. The 'Newton–Fourier' problem was another, and only 10 days after praising Kempe's proof he wrote again to the young man, but this time about mechanical linkage problems involving the analysis of systems of jointed rods, and the curves that they generate. Like many of the problems that Cayley addressed, the four-colour problem passed in and out of his attention. It was, however, a significant problem and we might have expected an entry in the *Jahrbuch die Fortscritte der Mathematik* referring to Kempe's proof in the *American Journal of Mathematics*. Cayley was one of the reporters for the abstracting journal, and given his praise of Kempe's paper it is odd that no statement appeared, not even a brief one. But his own paper in the Royal Geographical Society's *Proceedings* was not reviewed either, perhaps more understandably because it was published in a journal not normally surveyed by the *Jahrbuch*. The *American Journal of Mathematics* was in the cast list for the abstracting journal and, by any standards, Kempe's paper was a substantial one.[52]

With a few exceptions, the problem was not reported in the *Jahrbuch* at all, suggesting it to be a problem on the periphery of mathematics and not central to the mainstream (like invariant theory, for example). P. G. Tait learned of the problem from Cayley and 'solved it' (while invigilating an examination), although his proof turned out to suffer from the defect of circular reasoning (reasoning that assumes what it is hoped to be proved). Nevertheless, Cayley did report on Tait's first paper of 1880 with an innocuous one-line statement: 'Refers to De Morgan's theorem found empirically, that four colours are sufficient, in order that the different districts of a map can all be distinguished from each other.'[53] When Cayley visited Felix Klein in the summers of 1879 and 1880, he possibly interested him in the problem, and Klein later pointed out its connection with the problem of the five princes, which had been publicized in about 1840 by A. F. Möbius. In this, a kingdom is to be divided between five princes so that each part has a border in common with each of the other parts. Such a division of the kingdom is in fact not possible, and, contrary to the beliefs of some leading mathematicians, the problem of the five princes is not mathematically equivalent to the four-colour problem.[54]

After his brief connection with the four-colour map problem Cayley never again commented on it. When in the late 1880 s he began editing his former mathematical articles for inclusion in his *Collected Mathematical Papers* he did get as far as looking again at the one paper he published in the Royal Geographical Society *Proceedings*, and it was one that escaped the notice of his principal obituarist A. R. Forsyth.[55] It was, however, a paper that set the ball rolling on a tantalizing problem—one which has both teased and challenged mathematicians and given rise to a rich literature.

Acknowledgements

Quotations from the Kempe/Cayley letters have been made by courtesy of the representatives of the late Rev. A. H. M. Kempe, and the County Archivist of West Sussex. For permission to quote from the Klein/Cayley correspondence I thank the Department of Manuscripts, Niedersächsische Staats- und Universitätsbibliothek, Göttingen. For permission to quote from the Sylvester Papers, I am grateful to the Master and Fellows of St John's College, Combridge. The Royal Geographical Society gave permission to quote from correspondence and the Cayley draft manuscripts. Finally, I thank Dr Robin Wilson for his helpful comments on a preliminary draft of this paper.

Notes

1 Books devoted to the four-colour map problem include Robin J. Wilson, *Four colours suffice* (London: Allen Lane, 2002) (a historical approach); Robert A. Wilson, *Graphs, colourings and the four-colour theorem* (Oxford: Oxford University Press, 2002) (a technical account); Rudolf Fritsch and Gerda Fritsch, *The four-color theorem: history, topological foundations, and the idea of proof* (New York: Springer, 1998); David Barnette, *Map coloring, polyhedra, and the four-color problem* (New York: Mathematical Association of America, 1983); T. L. Saaty and P. C. Kainen, *The four-color problem: assaults and conquests* (London: McGraw-Hill, 1977; New York: Dover reprint, 1986); O. Ore, *The four-color problem* (New York: Academic Press, 1967). Bibliographic details to journal articles can be found in this literature.

2 Saaty and Kainen, *op. cit.* (note 1), p. 3.

3 Cayley's mathematical works are contained in *The collected mathematical papers of Arthur Cayley* (ed. A. Cayley and A. R. Forsyth), 13 volumes plus supplement (Cambridge University Press, 1889–98). This work is abbreviated here as *Coll. Math. Papers*.

4 A. Cayley, 'On the colouring of maps', *Proc. R. Geogr. Soc.* **1**, 259–261 (1879) and *Coll. Math. Papers* **11**, 7–8.

5 Saaty and Kainen, *op. cit.* (note 1), p. 6.

6 Drawing on A. B. Kempe's theory of 'mathematical form', A. Cayley advanced colour-related concepts (colourgroup, colourbond) in his extensive paper (A. Cayley, 'On the theory of groups', *Am. J. Math.* **11**, 139–157 (1889) and *Coll. Math. Papers* **12**, 639–656).

7 A. Cayley, 'On the geometrical representation of Cauchy's theorem of root-limitation', *Camb. Phil. Soc. Trans.* **12**, 395–413 (1877) and *Coll. Math. Papers* **9**, 21–39. Sable (black), gules (red), argent (silver or white) and azure (blue) are the names of tinctures that occur in heraldry.

8 Instances of Cayley's adoption of colour as a research aid occur in geometry, algebra and analysis are: A. Cayley, 'Notices of communications to the London Mathematical Society', *Proc. Lond. Math. Soc.* **2**, 6–7 (1866) and *Coll. Math. Papers* **6**, 19–20; 'On Dr Wiener's model of a cubic surface with 27 real lines; and on the construction of a double-sixer', *Camb. Phil. Soc. Trans.* **12**, 366–383 (1873) and *Coll. Math. Papers* **8**, 366–384; 'On a diagram connected with the transformation of elliptic functions', *Br. Assoc. Rep.* 534–535 (1881) and *Coll. Math. Papers* **11**, 26.

9 A. Cayley, 'A sixth memoir on quantics', *Phil. Trans. R. Soc. Lond.* **149**, 61–90 (1859) and *Coll. Math. Papers* **2**, 561–592.

10 Cayley identified the exceptions to the 'ordinary' $v+f=e+2$ formula as the small stellated dodecahedron ($m=1$, $n=2$, $D=3$) and the great dodecahedron ($m=2$, $n=1$, $D=3$). Cayley defined D, the density of a polyhedron, but did not give it this name in A. Cayley, 'On Poinsot's four new regular solids', *Phil. Mag.* **17**, 123–128 (1859) and *Coll. Math. Papers* **4**, 81–85. Two weeks after publishing this paper Cayley read Cauchy's 1810 writings on polyhedra with admiration and made an addendum in A. Cayley, 'Second note on Poinsot's four new regular polyhedra', *Phil. Mag.* **17**, 209–210 (1859) and *Coll. Math. Papers* **4**, 86–87. In general, see P. R. Cromwell, *Polyhedra* (Cambridge University Press, 1997), pp. 257–259.

11 A. Cayley, 'On the partitions of a close', *Phil. Mag.* **21**, 424–428 (1861) and *Coll. Math. Papers* **5**, 62–65 and 617.

12 Cayley, *op. cit.* (note 11), p. 63. A *close* takes its terminology from an enclosed space but it is not necessarily a bounded set in the modern sense. For Cayley the infinite surface of the plane is a close (being bounded by a contour (a closed curve not cutting or meeting itself) at infinity).

13 The concepts of path connectedness and topology of the plane were sketched out in two undated letter fragments: A. Cayley to J. J. Sylvester (1860), *Sylvester Papers*, St John's College, Cambridge, Box 2.

14 Cayley was one of the first to recognize the topological character of the 'Eulerian' polyhedral formulae, according to G. Burde and H. Zieschang, 'Development of the concept of a complex', in *History of topology* (ed. I. M. James) (Elsevier, Amsterdam, 1999), p. 104.

15 At the committee meeting, those present were (in order of seniority) Thomas Cotterill (70 years old), Arthur Cayley FRS (56), John James Walker FRS (52), Henry John Stephen Smith FRS (51), Charles Watkins Merrifield FRS (51), Thomas Archer Hirst FRS (48), Robert Tucker (46), Olaus Magnus Henrici FRS (38), Morgan Jenkins (37), James Whitbread Lee Glaisher FRS (29) and Alfred Bray Kempe (28). The president, John William Strutt FRS (35), was absent and the vice-president, Henry Smith, took the chair.

16 A. Cayley to F. Klein, 15 May 1878, Niedersächsische Staats- und Universitätsbibliothek, SUB Göttingen, Cod. Ms F. Klein 8: 365–391.

17 Cayley used the term 'hexahedron' for what is usually called a 'cube.' The octo-hexahedron formed by cutting off tetrahedra from each corner is one of the Archimedean solids now referred to as a truncated cube or cub-octahedron. It is quasi-regular, 'half-way between' the cube and the octahedron.

18 A. Cayley, 'On the theory of groups', *Proc. Lond. Math. Soc.* **11**, 126–133 (1878), p. 132, and *Coll. Math. Papers* **10**, 324–330.

19 The house at 22 Albemarle Street is adjacent to the Royal Institution of Great Britain, and was the London home of Florence Nightingale in the 1850s. A. Rice and R. J. Wilson, 'From national to international society: the London Mathematical Society 1867–1900', *Hist. Math.* **25**, 185–217 (1998).

20 The extra members included James Stirling (FRS 1902) (42 years old), Robert Forsyth Scott (28) and Charles Pendlebury (23). Thomas Robert Terry (29) was admitted to the society at the meeting.

21 For H. MacColl's life and work, see M. Astroh, I. O. Grattan-Guinness and S. Read, 'A survey of the life of Hugh MacColl (1837–1909)', *Hist. Philosophy Logic* **22**, 81–98 (2001).

22 London Mathematical Society, *Minutes of Meetings* 1875–1884.

23 Cayley's query also appeared in *Proc. London. Math. Soc*. **9** (1877–78), p. 148; *Nature* **18**, 294 (1878).

24 In K. Pearson. *The life, letters and labours of Francis Galton* (Cambridge University Press,) vol. 2, p. 1 (1924).

25 F. Galton, *Memories of my life* (London: Methuen, 1908), p. 72.

26 Galton, *op. cit.* (note 25), p. 72.

27 Pearson, *op. cit.* (note 24), vol. 2, p. 1.

28 Pearson, *op. cit.* (note 24), vol. 2, p. 21.

29 Pearson, *op. cit.* (note 24), vol. 2, p. 22, note 3.

30 J. J. Sylvester to Cayley, 21 Aug. 1878, Sylvester Papers, St John's College, Cambridge, Box 11.

31 Cayley's mathematical activity during 1878–79 resulted in about 81 items consisting of notes and short papers on such subjects as the double and triple theta functions as well as geometrical topics.

32 A. Cayley, 'On the regular solids', *Q. J. Pure Appl. Math*. **15**, 127–131 (1878) and *Coll. Math. Papers* **10**, 270–273. I place the writing of this paper around November/December 1878.

33 A. Cayley considered the colouring of faces of polyhedra in the course of his studies in group theory before his return to Cambridge in 1863 (A. Cayley, 'Notes on polyhedra', *Q. J. Pure Appl. Math*. **7**, 304–316 (1863) and *Coll. Math. Papers* **5**, 529–539. This involves one of the outstanding problems in combinatorial topology today: a leading result is that a trivalent polyhedron can be properly coloured with three colours if and only if each face has an even number of sides (see Cromwell, *op. cit.* (note 10), p. 341). For example, for the cube and the tetrahedron, both trivalent polyhedra, the cube can be coloured with three colours but the tetrahedron (whose faces have three sides) requires four. Stated in terms of maps, this result is contained in Kempe's 1879 paper as a special case (see note 45).

34 Henry Walter Bates FRS (1825–92), explorer of South America.

35 F. Galton to H. W. Bates, 24 Jan. 1879, Roy. Geog. Soc., JMS 19/24, Maps 1879.

36 A. Cayley to F. Galton, 23 Jan. 1879, Roy. Geog. Soc., JMS 19/24, Maps 1879.

37 A. De Morgan referred to the four-colour problem for a review in the *Athenaeum* (No. 1694, 14 April 1860, pp. 501–503). This reference was located by J. Wilson in 'New light on the origin of the four-colour conjecture', *Hist. Math*., **3**, 329 (1976).

38 This Smith's prize examination was sat by Karl Pearson at Cayley's house, in which Cayley's hospitality included not 'sparing his cellar.' For a history of the Smith's prize undergraduate examination, which students sat as an extra to the Tripos examinations, see J. Barrow-Green, '"A corrective to the Spirit of too Exclusively Pure Mathematics": Robert Smith (1689–1768) and his Prizes at Cambridge University', *Ann. Sci.* **56**, 271–316 (1999).

39 A. Cayley to W. Thomson, 7 Feb. 1879, Camb. Univ. Lib., Add 342.C62; A. Cayley to T. A. Hirst, 10 Feb.1879, Univ. Coll. Lib., Lond. Math. Soc. Archive.

40 A. Cayley's talk to the Cambridge Philosophical Society occurred on 24 February 1879; A. Cayley to J. J. Sylvester, 3 March 1879, Sylvester Papers, St John's College, Cambridge, Box 2.

41 Sir Archibald Geike, *A long life's work: an autobiography* (Macmillan, London, 1924), p. 339.

42 A. Cayley to A. B. Kempe, 24 June [1879], West Sussex Record Office (Chichester), Kempe/53/6/1–2. For maps drawn on the surface of a torus, seven colours are both necessary and sufficient, as shown by P. J. Heawood in 1890 (Robin J. Wilson, *op. cit.* (note 1), pp. 132–138).

43 *Proc. Lond. Math. Soc.* **10**, 229–231 (1878–79).

44 *Nature* **20**, 275 (1879).

45 A. B. Kempe's, paper is entitled 'On the geographical problem of the four colours', *Am. J. Math.* **2**, 193–200 (1879). A simplified version also appeared in Nature **21**, 399–400 (1880).

46 J. J. Sylvester to A. B. Kempe, 16 July 1879, West Sussex Record Office (Chichester), Kempe/13/25.

47 The *American Mathematical Journal* was an appropriate journal because it could be read by C. S. Peirce, who had been attempting to settle the four-colour map problem since the 1860s.

48 J. J. Sylvester to A. B. Kempe, 7 Sept.1879, West Sussex Record Office (Chichester), Kempe/13/26.

49 A. Cayley to A. B. Kempe, 22 Nov. 1879, West Sussex Record Office (Chichester), Kempe/13/15.

50 Royal Society Memb. Cert, 2 June 1881, vol. 11, Cert. 12 (EC/1881/13). The first seven papers of Kempe's cited in the certificate are on kinematics and linkwork, and the eighth is on the four-colour map problem.

51 It is likely that Sylvester went to his grave thinking Kempe's proof was correct. He wrote to Kempe eight months before his death: 'The problem of the 4 colors solved will ever be a bright leaf in your laurel crown' (J. J. Sylvester to A. B. Kempe, 5 June 1896, West Sussex Record Office (Chichester), Kempe/29/5).

52 The four-colour map problem was generally forgotten immediately after Kempe's 'proof'. It did have an airing in 1883 in the *Revue Scientifique* in which Edouard Lucas provided a French translation of Kempe's paper as pointed out by Robin J. Wilson (*op. cit.* (note 1), p. 235).

53 'Bezieht sich auf De Morgan's durch Erfahrung gefundenen Satz, dass vier Farben hinreichend sind, um auf einer Karte die verschiedenen Districte alle von einander zu unterscheiden' (*Jb. Fortschr. Math.* 12.0408.02). The paper Cayley reviewed in the *Jahrbuch über der Fortschritte der Mathematik* was P. G. Tait, 'On the colouring of maps', Proc. Roy. Soc. Edin. **10**, 501–503 (1880). Tait learned of the problem from Cayley (N. L. Biggs, E. K. Lloyd and R. J. Wilson *Graph theory 1736–1936* (Oxford, Clarendon Press, 1976; reprint 1998), p. 103.).

54 See Robin J. Wilson, *op. cit.* (note 1), pp. 34–37 and 96.

55 A. R. Forsyth, 'Arthur Cayley', *Proc. R. Soc. Lond.* **58**, 1–43 (1895), and *Coll. Math. Pap.* **8**, ix–xliv.